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AXISYMMETRIC ANTIDYNAMO THEOREMS IN COMPRESSIBLE NON-UNIFORM CONDUCTING FLUIDS

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CONTENTS

This paper considers magnetic fields generated by dynamo action in electrically conducting fluids under axisymmetric but otherwise very general non-steady conditions, including compressible flow $\boldsymbol{v}$, variable magnetic diffusivity, general volume shape (possibly even a union of disconnected conductors) with smooth possibly-moving boundary. We critically review previous antidynamo results and are forced to question a number of the more important ones.

Using operator inequalities and maximum principles for uniformly elliptic and parabolic partial differential equations we construct a number of functions ('comparison functions') that bound the meridional flux function χ. In particular, we derive the uniform bounds

$$|\chi| \leqslant \begin{cases} X(0), & 0 \leqslant t \leqslant \tau, \\ X(0)\,\dfrac{t}{\tau}\,\mathrm{e}^{1-t/\tau}, & t \geqslant \tau, \\ Y_1(0)\,F_1(\varpi)\,\mathrm{e}^{-t/\tau}, & t \geqslant 0, \end{cases}$$

where $X(t) = \max|\chi|$ and $Y_1(t) = \max|\chi/F_1(\varpi)|$ at time t; $F_1 \geqslant 0$ being a prescribed function of cylindrical radius ϖ; and equality in the first bound being only possible at $t = 0$. These bounds prove that χ decays uniformly in space and unconditionally to zero; by advancing the time origin, that $X(t)$, $Y_1(t)$ decay strictly monotonically to zero; and τ is an upper bound for the decay time. By using Schauder-type *a priori* estimates, spherical harmonic analysis and other techniques, it follows from these bounds that other field parameters, such as the meridional vector field $\boldsymbol{B}_{\mathrm{m}}$, the external multipole moments, the toroidal current density and the net outward surface flux, all decay to zero (but not all uniformly). The decay-time bound τ is directly related to the magnetic Reynolds number R (based on the speed and radius suprema, and diffusivity infimum). For free decay ($R = 0$) it is seen that variations in diffusivity and volume shape cannot extend the decay time by more than a factor of about 2.5 over the poloidal free-decay time in a fixed uniform sphere ($\tau_{\mathrm{pol}} = \pi^{-2}$). However, for large R, τ may take very large values (for example $\tau \gtrsim 10^{17}$ diffusion time units when $R \approx 10^2$).

The same comparison function approach applied to the azimuthal field parameter $A = B_\phi/\varpi$ leads to pointwise decay bounds analogous to those given above provided $R\nabla\cdot\boldsymbol{v}$ and $\partial\eta/\partial\varpi$ are small (see (5.16)); but, as for χ, the decay-time bounds may sometimes be exceedingly large. These decay bounds for A also assume that given the decay of $\boldsymbol{B}_{\mathrm{m}}$, the rate of generation, $(v_\phi \boldsymbol{B}_{\mathrm{m}} A)$ say, of A by differential rotation shearing the $\boldsymbol{B}_{\mathrm{m}}$-lines, is reasonably modelled by a uniformly decaying function. For the special case of free decay in a uniform fluid (where $\boldsymbol{v} = \partial\eta/\partial\varpi = (v_\phi \boldsymbol{B}_{\mathrm{m}} A) = 0$) it is seen that variations in volume shape cannot extend the free-decay time by more than a factor of about 2.5 over the toroidal free-decay time in a fixed uniform sphere ($\tau_{\mathrm{tor}} \simeq (1.43\pi)^{-2}$); and for a uniform incompressible fluid (where $\nabla\cdot\boldsymbol{v} = \partial\eta/\partial\varpi = 0$, but $(v_\phi \boldsymbol{B}_{\mathrm{m}} A) \not\equiv 0$) it is seen that A decays uniformly pointwise to zero. For compressible non-uniform fluids we prove more generally (i.e. regardless of $\nabla\cdot\boldsymbol{v}$ and $\partial\eta/\partial\varpi$) that $\|A\|_1$, the volume integral of $|A|$, cannot grow above a finite bound determined by $(v_\phi \boldsymbol{B}_{\mathrm{m}} A)$. When $(v_\phi \boldsymbol{B}_{\mathrm{m}} A)$ is negligible $\|A\|_1$ is shown to decay strictly monotonically. And if $\ln|A|$ does not develop negative ϖ-gradients steeper than $\ln(\mathrm{constant}/t^{\frac{1}{2}})$ for large t, then $\|A\|_1$ decays to zero beneath a comparison function, which again may decay extremely slowly.

One of our main conclusions is that axisymmetric antidynamo theorems allowing compressible flow in non-uniform fluids have not *yet* been shown to be generally effective, in the sense that they do not ensure decay on timescales that do not significantly exceed the relevant astroplanetary timescales, unless the compressibility and non-uniformities are specially restricted (as, for example, by Backus (*Astrophys. J.* **125**, 500 (1957))). Our most important results do not rely on any velocity boundary conditions and therefore apply directly to non-velocity mechanisms such as the Nernst–Ettingshausen thermomagnetic effect. The results herein include previously established antidynamo theorems as special cases, and the methods provide alternative proofs of, and strengthen, known steady antidynamo theorems.

1. Introduction

In 1919 Larmor (1919) suggested that the self-excited dynamo process might be responsible for the short- and long-term behaviour of cosmic magnetic fields. Fifteen years later Cowling (1934) published the first antidynamo theorem (a.d.t.) showing the impossibility of a steady axisymmetric dynamo, where both magnetic field and flow lines were in meridional planes. Since then generalizations have been made of the Cowling theorem and other a.d.ts have been discovered. Such theorems preclude in various senses the maintenance against ohmic decay of self-exciting fluid dynamos with certain symmetries or insufficiently 'vigorous flows' (see Moffatt (1978) for references). The usual proofs of several of these theorems, especially under non-steady conditions, depend critically on the flow $\boldsymbol{v}$ being incompressible, i.e. $\nabla \cdot \boldsymbol{v} \equiv 0$, a fact that has led Todoeschuck & Rochester (1980) to suggest that compressibility in the Earth and Saturn may be sufficient to vitiate the Cowling theorem. This suggestion highlights the need to determine what modifications must be made to a.d.ts in the presence of compressibility and otherwise variable conditions. If a small amount of compressibility does assist field maintenance, this may facilitate symmetric dynamo modelling of planetary and cosmic magnetic fields.

The present paper treats the axisymmetric problem as defined in §2. In §3 we review previous attempts to generalize the Cowling theorem to compressible flows and variable conductivity, and indicate that no such generalization has previously been completely achieved. The remainder, and main part, of the paper shows:

(i) the flux function χ of the magnetic field decays pointwise to zero regardless of compressibility, variable conductivity or permeability, volume-shape or moving boundaries;

(ii) decay to zero of related parameters, e.g. multipole moments and meridional vector field;

(iii) that the azimuthal component B_ϕ of the magnetic field decays pointwise to zero if the compressibility and conductivity or permeability variations are suitably restricted;

(iv) that the volume integral of $|B_\phi|/\varpi$, denoted $\|A\|_1$, cannot increase above a bound determined by the strength of the differential rotation and meridional magnetic field;

(v) that in the absence of differential rotation or a meridional field, then, $\|A\|_1$ decays; and:

(vi) determines seemingly reasonable conditions under which the decay of $\|A\|_1$ is to zero;

(vii) obtains upper bounds on decay times of the meridional and azimuthal fields.

This last point (vii) is an important, but usually neglected aspect of a.d.ts. For even if decay to zero is shown to occur, an a.d.t. does not rule out field maintenance over time spans of physical interest if the decay time is much greater than the characteristic time of the particular astroplanetary field being studied. It will be seen in this paper that axisymmetric a.d.ts have not yet been proven effective in the foregoing sense in the presence of compressibility.

Some of our main conclusions rely on the mathematical theories of maximum principles and *a priori* estimation for second order elliptic and parabolic partial differential inequalities and equations. As background references for these topics, we recommend the works of Protter &

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Weinberger (1967) (hereinafter P.W.; especially recommended for readers unfamiliar with maximum principles), Gilbarg & Trudinger (1977) and Friedman (1964). While maximum principles are sometimes couched in sophisticated mathematical jargon, they are basically simple implications of the usual properties of the derivatives of a function at its local maxima, and the relating of these derivatives via the differential equation or inequality.

2. Problem and field descriptions

2.1. *The non-uniform, compressible, axisymmetric induction problem*

We consider magnetic fields arising from electric currents inside a fluid moving with velocity field $\boldsymbol{v}$, having possibly non-uniform electrical conductivity σ, and occupying volume $V(t)$ at time t. V_∞ denotes all space; $\hat{V}(t)$ the exterior of $V(t)$; $S(t)$ the surface of $V(t)$ and $\boldsymbol{n}$ a unit outward normal. We use cartesian and polar coordinates related by $(x_i; i = 1,2,3) = (x,y,z) = (\varpi\cos\phi, \varpi\sin\phi, r\cos\theta)$, where $\varpi = r\sin\theta$. Unit reference vectors and components are denoted $\boldsymbol{e}_r$, v_r (polar), v_i (cartesian), etc. It will be important for applications of the theorems in §§4, 5 to pay particular attention to such things as the boundedness, continuity and differentiability of the functional models representing the various physical fields. Choosing appropriate non-negative constants $a, \eta_0, \eta_1, K_1, \ldots, K_6$, we adopt the following physically reasonable conditions.

(i) V and $\hat{V}$ are connected in the sense that any two points in V (or $\hat{V}$) are joinable by a curve lying everywhere inside $V(\hat{V})$. $\hat{V}$ is non-conducting and neutral. These assumptions are clearly true in many stellar and planetary applications where V is simply a sphere and $\hat{V}$ free space. (For generalizations, see §6, G.2, G.3, G.5.) V is confined to a finite region of space, say $0 \leqslant r \leqslant a$ in V for all t. S is not necessarily spherical, may be moving, but is smooth: class C^2 is sufficient (Friedman 1964, p. 86). The finite extent of V ensures, among other things, that $\hat{V}$ cuts the z-axis, extends to infinity, and that in quasistationary conditions

$$|\boldsymbol{B}| = O(r^{-3}) \quad \text{as} \quad r \to \infty. \tag{2.1}$$

(ii) The magnetic permeability μ is everywhere the free space value μ_0. (For generalization, see §6, G.1.)

(iii) V, σ, $\boldsymbol{v}$, $\boldsymbol{B}$ are axisymmetric. This requires, in particular, that

$$v_\phi = B_\phi = v_\varpi = B_\varpi = 0 \quad \text{at} \quad \varpi = 0. \tag{2.2}$$

(iv) The electrical conductivity σ may be discontinuous across S. But in V, the magnetic diffusivity $\eta = 1/\mu\sigma$ has continuous first derivatives in time and second derivatives in space, and

$$0 < \eta_0 \leqslant \eta \leqslant \eta_1 < \infty,$$

$$|\nabla\eta| \leqslant K_1\eta_0 a^{-1}, \quad |\partial\eta/\partial t| \leqslant K_2\eta_0^2 a^{-2}. \tag{2.3a, b}$$

Any axisymmetric scalar function that is differentiable throughout V must have zero ϖ-gradient on the z-axis. So, if $\partial^2\eta/\partial\varpi^2$ exists, then

$$\partial\eta/\partial\varpi = O(\varpi) \quad \text{as} \quad \varpi \to 0. \tag{2.4}$$

(Note that the same is not generally true for polar components of vectors. For example, B_ϕ is not necessarily even differentiable with respect to x or y at $\varpi = 0$.) We further assume that

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(2.4) holds uniformly in z and t. It then follows by conjunction with (2.3a) that there exists constant K_3 such that

$$|\partial\eta/\partial\varpi| \leqslant K_3\eta_0 a^{-2}\varpi \quad \text{in } V. \tag{2.5}$$

Equation (2.5) is used when discussing the decay of B_ϕ but is not needed when proving decay of the meridional components of $\boldsymbol{B}$.

(v) The flow $\boldsymbol{v}$ has continuous first space–time derivatives, and

$$|\boldsymbol{v}| \leqslant K_4, \quad |\nabla v_i| \leqslant K_4 K_5 a^{-1}, \quad |\partial v_i/\partial t| \leqslant K_4 K_6 \eta_0 a^{-2}. \tag{2.6a, b, c}$$

(vi) Surface currents are not present, and consistent with (ii), $\boldsymbol{B}$ is therefore continuous across S. Additional to differentiability inherent in the pre-Maxwell equations (vii) we assume only that the second cartesian derivatives of $\boldsymbol{B}$ exist continuous in V. This is a weaker-than-normal assumption, used in (vii) and §2.2.

(vii) Consistent with condition (ii), the electromagnetic field is governed by the quasi-stationary pre-Maxwell equations

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{j}, \quad \nabla\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t, \quad \nabla\cdot\boldsymbol{B} = 0, \tag{2.7$a, b, c,$}$$

and the Ohm law

$$\boldsymbol{j} = \sigma(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}). \tag{2.8}$$

Here, $\boldsymbol{j}$ is the current density and $\boldsymbol{E}$ the electric field. Equations (2.7), (2.8) and condition (vi) lead to the induction equation in V,

$$\partial\boldsymbol{B}/\partial t = \nabla\times(-\eta\nabla\times\boldsymbol{B}+\boldsymbol{v}\times\boldsymbol{B}), \tag{2.9}$$

which allows for variable η; and the current-free condition in $\hat{V}$:

$$\nabla\times\boldsymbol{B} = 0. \tag{2.10}$$

We introduce the dimensionless variables

$$r' = r/a, \quad 0 \leqslant r' \leqslant 1 \quad \text{in } V, \tag{2.11a}$$

$$\varpi' = \varpi/a, \quad 0 \leqslant \varpi' \leqslant 1 \quad \text{in } V, \tag{2.11b}$$

$$t' = \eta_0 t/a^2, \quad \text{the diffusion timescale,} \tag{2.11c}$$

$$\boldsymbol{v}' = \boldsymbol{v}/K_4, \quad 0 \leqslant |\boldsymbol{v}'| \leqslant 1 \quad \text{in } V, \tag{2.11d}$$

$$\eta' = \eta/\eta_0, \quad 1 \leqslant \eta' \leqslant \eta_1/\eta_0 \quad \text{in } V; \tag{2.11e}$$

and hereinafter suppress the primes. With this scaling, constraints (2.3), (2.5), (2.6) simplify to the dimensionless forms

$$|\nabla\eta| \leqslant K_1, \quad |\partial\eta/\partial t| \leqslant K_2; \tag{2.12a, b}$$

$$|\partial\eta/\partial\varpi| \leqslant K_3\varpi \quad \text{in } V; \tag{2.13}$$

$$|\nabla v_i| \leqslant K_5, \quad |\partial v_i/\partial t| \leqslant K_6. \tag{2.14a, b}$$

As already indicated, not all of conditions (i) to (vii) are necessary. Generalizations of lesser interest will be discussed in §6. Our main aim in this paper is to determine whether or not the $\boldsymbol{B}$ defined above, and various parameters derived from $\boldsymbol{B}$, decay to zero; and if so, what bounds can be put on the decay rate.

2.2. *The meridional field*

If $\boldsymbol{B}_{\mathrm{m}}$ is the meridional component of $\boldsymbol{B}$, then (2.7*c*) and the axisymmetry of $\boldsymbol{B}$ imply $\nabla\cdot\boldsymbol{B}_{\mathrm{m}} = 0$. This allows introduction of the flux function $\chi(\varpi, z, t)$ such that

$$\boldsymbol{B}_{\mathrm{m}} = \nabla\times(\chi\boldsymbol{e}_{\phi}/\varpi). \tag{2.15}$$

By the Stokes theorem $2\pi\chi(\varpi, z, t)$ is the flux of $\boldsymbol{B}_{\mathrm{m}}$ or $\boldsymbol{B}$ through any surface capping the azimuthal circle through (ϖ, z), provided χ is adjusted to vanish at $\varpi = 0$. Consistent with conditions (ii) and (iv) (but, more generally, see §6, G.1), the first derivatives of χ are continuous in V_{∞}, and the second derivatives in V, $\hat{V}$, but not across S in general. Condition (vi), and (2.1), (2.2) for B_{ϖ}, and (2.15), imply

$$\chi = O(r^{-1}) \quad \text{as} \quad r\to\infty; \qquad \chi = O(\varpi^2) \quad \text{as} \quad \varpi\to 0. \tag{2.16\textit{a, b}}$$

The ϕ-components of (2.7*a, b*) and (2.8), scaled according to (2.11), show that χ satisfies

$$\mathscr{P}\chi = 0 \quad \text{in } V, \tag{2.17}$$

where

$$\mathscr{P} \equiv \eta\nabla^2 - \left(\frac{2\eta}{\varpi}\boldsymbol{e}_{\varpi} + R\boldsymbol{v}\right)\cdot\nabla - \frac{\partial}{\partial t}, \tag{2.18}$$

and $R = aK_4/\eta_0$ is the magnetic Reynolds number based on the speed supremum; and, from the ϕ-component of (2.10), χ satisfies

$$\mathscr{E}\chi = 0, \quad \text{in } \hat{V}, \tag{2.19}$$

where

$$\mathscr{E} \equiv \nabla^2 - \frac{2}{\varpi}\frac{\partial}{\partial\varpi}. \tag{2.20}$$

$\mathscr{P}$ and $\mathscr{E}$ are parabolic and elliptic differential operators, respectively. The property $\eta \geqslant 1$ (2.11*e*) is important in that it permits use of theorems for *uniformly* or *strictly*, or both, elliptic and parabolic operators (P.W.; Gilbarg & Trudinger 1975; Friedman 1964).

2.3. *The azimuthal field*

If $\boldsymbol{B}$ is differentiable throughout V, then axisymmetric condition (iii) (2.2) for B_{ϕ} implies $B_{\phi} = O(\varpi)$ as $\varpi\to 0$. Furthermore, if $\boldsymbol{B}$ is twice differentiable, as in condition (vi), then application of condition (iii) to the relation between $\partial^2 B_{\phi}/\partial\varpi^2$ and the second cartesian derivatives of $\boldsymbol{B}$, implies $\partial^2 B_{\phi}/\partial\varpi^2 = 0$ at $\varpi = 0$. And near $\varpi = 0$,

$$B_{\phi}(\varpi, z) = A_1(z)\,\varpi + A_2(\varpi, z)\,\varpi^2, \tag{2.21}$$

where $A_1 = \partial B_{\phi}/\partial\varpi$ at $\varpi = 0$, and $2A_2 = \partial^2 B_{\phi}/\partial\varpi^2$ at some radius between 0 and ϖ. Equation (2.21) allows us to define $A = B_{\phi}/\varpi$ with the property

$$\left(\frac{\partial A}{\partial\varpi}\right)_{\varpi=0} = \lim_{\varpi\to 0}\left[\frac{A(\varpi, z) - A_1(z)}{\varpi}\right] = A_2(0, z) = 0. \tag{2.22}$$

The usefulness of (2.22) will be appreciated in §5.2. It is a slightly more general property than that noted by Backus & Chandrasekhar (1956), who assumed $\boldsymbol{B}$ *thrice* differentiable, leading to the stronger property $\partial A/\partial\varpi = O(\varpi)$ as $\varpi\to 0$.

The ϕ-component of the induction equation (2.9) shows that in V,

$$(\mathscr{Q}+c)\,A = (v_\phi\, \boldsymbol{B}_{\mathrm{m}}\, A), \tag{2.23}$$

where
$$\mathscr{Q} \equiv \eta\nabla^2 + \left(\frac{2\eta}{\varpi}\boldsymbol{e}_\varpi + \nabla\eta - R\boldsymbol{v}\right)\cdot\nabla - \frac{\partial}{\partial t} \tag{2.24}$$

is a uniformly parabolic differential operator;

$$c \equiv \frac{2}{\varpi}\frac{\partial\eta}{\partial\varpi} - R\nabla\cdot\boldsymbol{v}, \tag{2.25a}$$

and
$$(v_\phi\,\boldsymbol{B}_{\mathrm{m}}\,A) = -R\boldsymbol{B}_{\mathrm{m}}\cdot\nabla(v_\phi/\varpi). \tag{2.25b}$$

Equation (2.23) may be written in divergence form:

$$\nabla\cdot\left[\frac{\eta}{\varpi^2}\nabla(\varpi^2 A) - R\boldsymbol{v}A + R\frac{v_\phi}{\varpi}\boldsymbol{B}_{\mathrm{m}}\right] - \frac{\partial A}{\partial t} = 0. \tag{2.26}$$

The general solution of the meridional component of (2.10) is $A = f(t)/\varpi^2$. Since $\hat{V}$ is connected and intersects the z-axis (§2.1 (i)) and A is continuous across S (§2.1 (vi)), it follows that

$$A \equiv 0 \quad \text{in} \quad \hat{V} \quad \text{and on} \quad S. \tag{2.27}$$

The source terms cA and $(v_\phi\,\boldsymbol{B}_{\mathrm{m}}\,A)$ make (2.23) manifestly different from (2.17) for χ. The term cA arises because of compressibility and non-uniform diffusivity. We will later require that c be bounded. In fact, with the space–time inf and sup of c denoted by c_{i} and c_{s}, it follows from (2.13) that

$$c_{\mathrm{s}} \leqslant 2K_3 + 3RK_5, \quad c_{\mathrm{i}} \geqslant -2K_3 - 3RK_5. \tag{2.28a, b}$$

The term $(v_\phi\,\boldsymbol{B}_{\mathrm{m}}\,A)$ represents the rate of generation of azimuthal field from meridional field by differential rotation. Apart from the considerations of Childress (1969), this coupling term appears to have been neglected by other authors of a.d.ts. Its presence will be allowed for, as far as possible, in §5.

3. Brief review of previous results

The essence of the Cowling theorem is that self-exciting axisymmetric dynamos cannot be maintained no matter how much energy is available in the velocity field. The word 'theorem' is often used loosely in a.d.t. contexts: in reality, the Cowling theorem embodies several theorems, proofs and part proofs, with assumptions and definitions of field maintenance that differ from author to author. The style and terminology of proof also varies, some authors using 'energy-type' integral methods, others mathematical properties of partial differential equations, and others arguments about the neutral lines (n.l.) of $\boldsymbol{B}_{\mathrm{m}}$ (i.e. where $\boldsymbol{B}_{\mathrm{m}} = \boldsymbol{0}$). In his original paper, Cowling (1934) used both a neutral line neighbourhood (n.l.n.) argument and a very simple maximum principle (m.p.) for elliptic partial differential equations. Whilst n.l. and m.p. proofs differ in mathematical emphasis, they are otherwise closely related. For, as is easily shown from (2.15), the maxima and minima of χ are O-type neutral points of $\boldsymbol{B}_{\mathrm{m}}$ and vice versa. (X-type neutral points correspond to saddle points.)

The question of whether a Cowling-like theorem holds independently of variable conductivity or independently of compressibility is not new. Several definite results are known. In particular, in steady conditions, when a.d.ts are equivalent to mathematical uniqueness, the only solution of the problem defined in §2 (and variants thereof) is $\boldsymbol{B} = \mathbf{0}$ (Cowling 1934; Backus & Chandrasekhar 1956; Lortz 1968, and see §4.2.3 and §5.4 herein). Even in this steady case the trail is confused by incomplete arguments, and conclusions that have waited to be substantiated by later authors. For a detailed historical critique, see Ivers (1984). Here we deal in detail only with the non-steady problem from which corresponding steady results follow as special cases.

Results are considerably less complete in non-steady conditions. Backus (1957) used maximum principles to show that irrespective of variable σ or compressible flow, the maximum (minimum) of χ cannot increase (decrease) above (below) its initial value. A roughly equivalent n.l. version of this same result has been given more recently by Parker (1979). The result is most easily derived from (2.7 *a*), (2.15), (2.17) and (2.19), and holds for all local maxima and minima of χ, as shown below.

Suppose χ_0 is a local maxima of χ occurring on an O-type neutral line L_0 in V. (As in arguments (i) and (ii) of §4.1, it can be shown that the n.ls of χ cannot occur in $\hat{V}$ or on S except in the trivial case $\chi \equiv 0$ in V_∞.) Then $(\nabla\chi)_0 = \mathbf{0}$ and $(\nabla^2\chi)_0 \leqslant 0$, so that by (2.17)

$$\partial\chi_0/\partial t = (\eta\nabla^2\chi)_0 \leqslant 0. \tag{3.1}$$

(Note that because $(\nabla\chi)_0 = \mathbf{0}$, $\partial\chi_0/\partial t$ is the derivative following the motion of L_0.) Hence χ_0 can never increase; and similarly no local minimum can decrease in time.

Parker (1979) argues further, although without detailed explanation, that when σ is constant (and finite) the magnetic field cannot increase faster than linearly with distance away from L_0, and hence that the n.l. current density

$$(j_\phi)_0 = -(\nabla^2\chi/\mu_0\varpi)_0 \tag{3.2}$$

can never be zero. If one accepts this argument or simply assumes that the result is true, even for non-uniform σ, then (3.1) implies that $\partial\chi_0/\partial t < 0$, i.e. that all local maxima and minima of χ must strictly decrease in magnitude as time increases.

The rationality of assuming that $(j_\phi)_0$ is non-zero is not entirely obvious. The contradictory possibility of higher order O-type neutral lines, where the gradients of $\boldsymbol{B}_{\mathrm{m}}$, and hence j_ϕ, vanish as well as $\boldsymbol{B}_{\mathrm{m}}$, has concerned other researchers and close examination shows that it can be allowed for in steady conditions by modifying the original Cowling neutral line neighbourhood (n.l.n.) proof (Cowling 1934; Moffatt 1978; Ivers 1984). It is important to distinguish between the Cowling n.l.n. proof, which applies the Stokes circulation theorem to a small meridional disc orthogonally transecting L_0 (a longitudinal circle), and the non-steady n.l. arguments, which apply the Stokes theorem to the 'horizontal' disc with boundary L_0 (compare discs C_ϵ and C_N in figure 6.1 of Moffatt (1978)). The n.l.n. argument works in steady conditions since $E_\phi = 0$ then; the non-steady n.l. argument works only with the added proviso that $(j_\phi)_0 \neq 0$, or at least that $(j_\phi)_0$ is not permanently zero after any finite time. A quirk of purely n.l. arguments is that in steady conditions, (3.1) leads *only* to the condition that $(j_\phi)_0 = 0$, not to the more complete result $j_\phi \equiv 0$. Supplementary arguments such as n.l.n. or m.ps are needed to show $j_\phi = 0$ away from the n.l.

An interesting variation on the n.l. method for the non-steady problem has recently been initiated by Hide (1979, 1981). This variation concerns the topology of the null-flux curves (n.f.c.) of $\boldsymbol{B}$, i.e. those closed lines on S on which $\boldsymbol{B}\cdot\boldsymbol{n} = 0$. Hide conjectures that dynamo action is impossible if, as is true in axisymmetry, there exists an axis passing through all the null-flux curves. As a basis for this conjecture, several attempts have been made to incorporate the properties of the null-flux curves into a proof of the non-steady axisymmetric a.d.t. A first attempt (Hide 1979) assumed that the current direction on the n.f.c. was in a right-handed sense relative to the local **B**-lines. This assumption is clearly not generally true since the behaviour of **B** near a n.f.c. can be dictated by the current density far from the n.f.c. Recognizing this shortcoming, Hide (1981) introduced certain 'control surfaces', the main effect being to replace the n.f.cs by n.ls of both O- and X-types. However, this novel approach does not seem to fully account for the contribution of the X-type n.ls and, for the O-type n.ls, must again assume $(j_\phi)_0 \neq 0$, as in the other non-steady n.l. proofs.

As an alternative approach, Hide & Palmer (1982) have more recently attempted to extend the Cowling n.l.n. proof to the non-steady case. To this end, (2.17) is integrated over a n.l.n., which is small enough for the diffusion term $\eta\nabla^2\chi$ to dominate the advection term $R\boldsymbol{v}\cdot\nabla\chi$ throughout. An argument is then made that if the n.l. current remains zero, then χ becomes undifferentiable as $t \to \infty$, which is a 'contradiction'. However, a requirement that χ be differentiable at $t = \infty$ would seem to be a constraint imposed additionally to the fundamental electromagnetic equations. (Many physically useful well behaved functions, for example $(1+x^2t^2)^{-1}$, have discontinuous limits.) It would be preferable if it could be shown that χ became undifferentiable in a time span of physical relevance, i.e. in a finite time.

It is unnecessary to make special assumptions concerning $(j_\phi)_0$ or $t = \infty$. Such problems can be overcome mathematically by using modern versions of m.ps, in ways similar to Lortz & Meyer-Spasche (1981), who showed that

$$X(t) = \max_{V_\infty} |\chi|$$

is a strictly monotonically decreasing function of time. This result is also a simple special case of the comparison function method used later herein (§4.2). However, even if one accepts that $X(t)$ decreases, none of the aforementioned arguments show that $X(t)$ decays to zero. In considering the behaviour of $X(t)$, note that the location (which may not be unique) of $X(t)$ may change discontinuously as the role of absolute maximum passes from one relative maximum of $|\chi|$ to another. According to (3.1) this can only occur by $X(t)$ decreasing to some existing relative maximum of $|\chi|$, not by any relative maximum of $|\chi|$ increasing. Correspondingly, $X(t)$ is possibly only piecewise continuously differentiable, and until proven otherwise $X(t)$ may decay towards a non-zero value, with the remainder of χ evolving underneath. Smaller relative maxima (minima) of χ may occasionally appear due to χ falling (rising) at surrounding points. These relative extrema may approach a limit (not yet shown to be zero), or just disappear, but when present must always decay apart from instantaneous inflexions.

One might argue that (3.1) and (3.2) incorporate the symbiotic nature of currents and fields and do indicate, in a loose sense, that the rate of decay of the field is dependent on the field strength, and therefore the decay of $X(t)$ must be zero. Such reasoning is far from sound. It is true that the field near L_0 determines the current there according to (3.2), but the converse is false: the field near L_0 is determined by the entire current distribution, and a more global argument is required. We doubt whether any argument to establish decay of χ can be local in the extreme sense of a purely n.l. argument.

If the right side of (3.1) contained an undifferentiated term $-p\chi_0$, then one could immediately infer that χ_0 decayed to zero at least as fast as $\exp(-pt)$. This style of argument has been previously used (Ivers & James 1981) to place an upper bound of five free decay times on the decay of a general (i.e. non-axisymmetric) poloidal field supported by spherically symmetric radial motions. When no undifferentiated term is present in (3.1), the task of bounding the decay (or growth) time is more difficult. Hide & Palmer (1982) argue, using order of magnitude estimates of (3.1), that χ decays to zero on the diffusion timescale. This argument disregards the behaviour of χ away from the n.l., and seems only to be valid if χ decays to zero away from the n.l. at least as fast as the free decay rate. To highlight the counter-possibility, suppose χ satisfies (3.1) throughout a disc of radius ϵ transecting the n.l. and $\chi = \exp(-pt)$ (for simplicity) on the boundary of the disc at time t. Then χ will decay to zero inside the disc provided $p > 0$. But if $p \leqslant 0$, χ will not decay to zero. In any case the long-term behaviour of χ is determined by the scale-time $\max\{1/|p|, \epsilon^2/\eta\}$, not just the diffusion time ϵ^2/η; and p is determined by both fluid motion and diffusion away from the n.l.

By using integral methods, several results associated with decay to zero have been established under restricted conductivity–compressibility conditions. In the simplest case of constant σ and incompressible flow the proofs of Cowling (1957) and Braginskii (1964) show strictly monotonic decay of

$$\|\chi\|_2 = \left(\int_V \chi^2 \,\mathrm{d}V\right)^{\frac{1}{2}}. \tag{3.3}$$

This result may be extended to *decay to zero* by using variational inequalities in a similar way to Braginskii (1964), who remarks in a footnote that the decay time when V is a sphere, can be shown to be no longer than the poloidal free-decay time $\tau_{\text{pol}} = \pi^{-2}$. The preciseness of this bound τ_{pol} has not been proven, and it seems that the optimum bound must lie between τ_{pol} and the longer bound $4\tau_{\text{pol}}$ given by Backus (1957). Backus *has* proven strictly monotonic decay to zero of

$$\|\rho^{\frac{1}{2}}\chi\|_2 = \left(\int_V \rho\chi^2 \,\mathrm{d}V\right)^{\frac{1}{2}},$$

where ρ is the mass density, with a decay time no longer than $4\tau_{\text{pol}}$. Backus' result is important in that it holds not only for uniform incompressible fluids, but also for variable σ and compressible flow, albeit with the assumptions that:

(i) V is a sphere;

(ii) ρ and $k = \rho\eta$ are spherically symmetric, i.e. dependent on r, t only;

(iii) $-4k < r\,\mathrm{d}k/\mathrm{d}t \leqslant 0$;

(iv) $3k + \mathrm{d}k/\mathrm{d}r > 0$ on S.

Backus argued that these conditions were approximately valid for the Earth. Under the same assumptions Backus (1957) also showed that the decay time of the external dipole moment M_1 was again no more than $4\tau_{\text{pol}}$, in the sense that

$$\lim_{t\to\infty} \mathrm{e}^{\lambda t}\int_t^\infty M_1 \,\mathrm{d}t = 0$$

for all $\lambda < 1/4\tau_{\text{pol}}$. The bound $4\tau_{\text{pol}}$ seems to definitely contradict the earlier numerical result of Chandrasekhar (1956) that even for an incompressible fluid with constant σ, fluid motion might extend the decay time by a factor of 20 or more (see Backus (1957), p. 503).

As mentioned in §2, the coupling term $(v_\phi \boldsymbol{B}_{\mathrm{m}} A)$ has usually been neglected in previous considerations of the azimuthal component of $\boldsymbol{B}$. When this is done in a uniform incompressible fluid, certain decay results can definitely be established from (2.23) (now with $c = (v_\phi \boldsymbol{B}_m A) = 0$). First, Backus (1957) has constructed an ingenious temperature analogue. Backus concludes that $\max\{|A|\}$ in $V(t)$ decays strictly monotonically to zero with decay time no greater than the poloidal free-decay time $\tau_{pol} = \pi^{-2}$. (There are several errors in Backus' arguments (on his pp. 515–517), but happily the proof seems repairable (Backus, personal communication) and his decay time conclusion valid. For example, Backus' equation (40) omits a non-trivial line integral along the z-axis, but in retrospect this integral can be shown to be always non-positive, a property that allows successful modification of Backus' ensuing arguments.) Also, as for χ, certain integral results are definitely established. The proofs of Cowling (1957) and Braginskii (1964) (also corrected by addition of a non-positive axis integral) show that

$$\|A\|_2 = \left(\int_V A^2 \, \mathrm{d}V\right)^{\frac{1}{2}}$$

decays strictly monotonically. Again, variational inequalities allow strengthening of this result to *decay to zero*, and thus Braginskii (1964) has remarked, again without proof, that when V is a sphere the decay time of $\|A\|_2$ is no longer than the toroidal free-decay time $\tau_{tor} \approx (1.43\pi)^{-2}$ (1.43π being approximately the first positive root of the spherical Bessel function $j_1(x)$). It seems probable that the correct optimum decay bound lies somewhere between the bounds τ_{tor} of Braginskii and τ_{pol} of Backus. The only detailed consideration of the influence of $(v_\phi \boldsymbol{B}_m A)$ (still in a uniform incompressible fluid) appears to be that of Childress (1969), who used Schwarz and variational inequalities in an attempt to modify the Braginskii analysis. Childress concludes that $\|A\|_2$ decays (not necessarily monotonically) to zero on the τ_{pol} timescale, while assuming there exists constant k_3 such that

$$\int_V |(v_\phi \boldsymbol{B}_m A)| \, \mathrm{d}V \leqslant k_3 \, \|\chi\|_2 \, \|A\|_2.$$

Given the formulae (2.15) and (2.25*b*) for $(v_\phi \boldsymbol{B}_m A)$, the existence of k_3 seems equivalent to bounding $\|\nabla\chi\|_2$ by $\|\chi\|_2$; which is the adverse of the usual variational relation, $\|\chi\|_2 \leqslant k_3' \, \|\nabla\chi\|_2$, and not generally possible. (For example, when χ is the nth degree free-decay mode in a sphere, it may be shown that $\|\nabla\chi\|_2/\|\chi\|_2 \to \infty$ as $n \to \infty$.) However, Childress' conclusion *is* correct and may be established without assuming the existence of k_3 (Ivers 1984).

Evidence against maintenance of A by compressible flows is much less substantial. Even if neglect of $(v_\phi \boldsymbol{B}_m A)$ is reasonable, (2.23) still contains the undifferentiated term cA, which, as simple heat flow analogies illustrate, may act as a source term. *Pointwise* decay or otherwise of A must be expected to depend on c. When $(v_\phi \boldsymbol{B}_m A)$ is omitted and η is constant, (2.23) is equivalent to equation (18.4) of Parker (1979), who, in spite of the presence of cA, describes the equation as source-free, and very heuristically argues for the decay of B_ϕ. Parker (1979) also claims the volume integral of A/ρ decays, but this does not seem to follow directly from his equation (18.4) unless ρ is constant (in which case c is zero) and A non-negative. More recently, for a non-uniform compressible fluid, Lortz & Meyer-Spasche (1982*b*) conclude that 'the toroidal field cannot grow in time'. What Lortz and Meyer-Spasche actually prove is that if V is convex (in the sense that $\partial\varpi/\partial n \geqslant 0$ on S), and if the generation term $(v_\phi \boldsymbol{B}_m A)$ is absent, then

$$\|A\|_1 \leqslant P(t), \tag{3.4}$$

where $\|A\|_1$ denotes $\int_V |A|\,\mathrm{d}V$, and $P(t)$ is some undetermined monotonically decreasing positive function. Care must be taken with the interpretation of (3.4). Unless one can show further that $P(\infty) = 0$, or that the two sides of inequality (3.4) are equal at some finite t, then the most one should conclude is that $\|A\|_1$ does not *grow without bound*, and that $\|A\|_1 \leqslant P(\infty)$ at $t = \infty$. If V is not convex (for example, a torus) the Lortz and Meyer-Spasche derivation of (3.4) requires modification as indicated by (5.30) in §5.5.

As indicated by the previous discussion, various senses of decay may be contemplated, and one sense does not necessarily imply another. Indeed, it is not even obvious that the decay of χ to zero implies the decay of $\boldsymbol{B}_\mathrm{m}$ (which depends on $\nabla\chi$) or other field parameters. (The expression $\exp(-t)\sin\{\exp(x+t)\}$ is a simple example of a function that decays to zero, but has a space gradient that does not.) Such problems are the basis of the 'spiky' and 'irregular' fields contemplated in figure 1 of Backus (1957) and perhaps on p. 319 of Cowling (1955).

Thus various questions have yet to be answered. (*a*) Does χ decay to zero? (*b*) If so, how can it be proven that $\boldsymbol{B}_\mathrm{m}$ and other meridional field parameters, such as external multipole moments and energy, decay to zero? (*c*) Does $\boldsymbol{B}_\phi$ decay to zero? (*d*) What are the decay rates? (*e*) Can the coupling term $(v_\phi \boldsymbol{B}_\mathrm{m} A)$ be rigorously accounted for? (*f*) Is spiky behaviour possible?

We will consider all of these questions in the following sections. Our basic approach for the meridional field will be to show that χ is dominated by specially constructed 'comparison functions', which decay uniformly to zero in time. This method was suggested by a decay theorem for certain parabolic boundary value problems on finite domains proven by Friedman (1964). The results have immediate temperature problem analogues and can be extended to more general convective–diffusive problems involving elliptic and parabolic differential equations in adjoining domains. The problem for the azimuthal field is simpler in that the domain V is finite, but complicated by the presence of the undifferentiated term cA and the coupling term $(v_\phi \boldsymbol{B}_\mathrm{m} A)$ in (2.23). It seems necessary to at least partially use an integral method that can take into account the divergence form of (2.26). With the principal exception of §5.5, we avoid using boundary conditions on $\boldsymbol{v}$, so that most of our results apply to non-velocity mechanisms such as the Nernst–Ettingshausen effect (see also Ivers & James (1981)).

4. Decay of the meridional field

4.1. *Comparison theorem for the flux function* χ

Decay to zero of the flux function χ is proven by constructing a 'comparison function' u that decays uniformly to zero in V_∞ and such that $|\chi| \leqslant u$. We will assume, to begin with, that u has continuous first derivatives in V_∞, except possibly for the z-axis; and continuous second derivatives in V_∞, except possibly for the z-axis and S. These conditions may be relaxed as, for example, in extensions E.3, E.4, which follow the theorem.

Theorem. *If, with* $\mathscr{E}$ *and* $\mathscr{P}$ *as in* (2.20), (2.18),

$$\mathscr{E}u \leqslant 0 \quad \text{in } \hat{V}; \tag{4.1}$$

$$\mathscr{P}u \leqslant 0 \quad \text{in } V; \tag{4.2}$$

$$u \geqslant |\chi| \quad \text{at} \quad t = 0; \tag{4.3}$$

$$u \geqslant 0 \quad \text{as} \quad \varpi \to 0 \quad \text{or} \quad r \to \infty; \tag{4.4}$$

then $u \geqslant |\chi|$ *in* V_∞ *for* $t \geqslant 0$.

Proof. Let the space supremum of $\chi-u$ at time t be $\mathscr{M}(t)$. Most of the following proof is concerned with showing that $\mathscr{M}(t) \leqslant 0$ for all $t \geqslant 0$.

If $\mathscr{M}$ occurs at $\varpi = 0$ or $r = \infty$, then $\mathscr{M} \leqslant 0$ by (2.16) and (4.4). Otherwise, $\mathscr{M}$ is just the space maximum of the continuous function $\chi-u$ and occurs on some time-dependent 'neutral' (i.e. maximal) circle N_0 in V, $\hat{V}$ or on S. Consider these three possibilities.

(i) N_0 lies in $\hat{V}$, $\varpi \neq 0$, $r \neq \infty$. By relations (2.19), (4.1) and the linearity of $\mathscr{E}$,

$$\mathscr{E}(\chi-u) \geqslant 0 \quad \text{in } \hat{V}. \tag{4.5}$$

By a maximum principle for elliptic inequalities (P.W., theorem 5, p. 61), $\chi-u \equiv \mathscr{M}$ throughout $\hat{V}$, in which case $\mathscr{M} \leqslant 0$ by (4.4) and (2.16). (Note that $\hat{V}$ is connected and must cut the z-axis by §2.1 (i).)

(ii) N_0 lies on S, $\varpi \neq 0$. By a second maximum principle for elliptic inequalities (P.W., theorem 7, p. 65) applied to (4.5), either (*a*) $\chi-u$ is strictly decreasing at N_0 in any direction into $\hat{V}$, in particular

$$\partial(\chi-u)/\partial n < 0 \quad \text{on } N_0, \tag{4.6}$$

or (*b*) $\chi-u \equiv \mathscr{M}$ in $\hat{V}$. Case (*a*) is impossible since, by the continuity of $\nabla(\chi-u)$, there would exist points near N_0 but inside V, where $\chi-u$ exceeded $\mathscr{M}$, contradicting the definition of $\mathscr{M}$. So only (*b*) is possible, in which case $\mathscr{M} \leqslant 0$ by (2.16) and (4.4).

To summarize arguments (i) and (ii): if $\mathscr{M}$ is attained in $\hat{V}$ or on S where $\varpi \neq 0$, then it is attained at all points of $S \cup \hat{V}$, and $\mathscr{M} \leqslant 0$ by (4.4) and (2.16).

(iii) The only remaining possibility is that N_0 is in V, where relations (2.17), (4.2) and the linearity of $\mathscr{P}$ imply

$$\mathscr{P}(\chi-u) \geqslant 0. \tag{4.7}$$

Since N_0 is strictly inside V,

$$\nabla(\chi-u) = 0 \quad \text{and} \quad \nabla^2(\chi-u) \leqslant 0$$

on N_0. Therefore, by direct substitution into (4.7),

$$\partial(\chi-u)/\partial t \leqslant 0 \quad \text{on } N_0. \tag{4.8}$$

The location of N_0, like that of L_0 in §3, may change with time, perhaps even discontinuously. But for situations of physical interest $\mathscr{M}(t)$ will presumably be at least piecewise differentiable with respect to t. With this presumption – see remark R.2 following, otherwise – (4.8) implies that $\mathscr{M}(t)$ is a non-decreasing function of t while N_0 is in V. The non-positivity of $\mathscr{M}(t)$ can then be simply determined by tracing the location of N_0 backwards in time. Two possibilities must be considered. (*a*) N_0 is in V at $t = 0$. Then $\mathscr{M}(0) \leqslant 0$ by (4.3), (*b*) N_0 moves into V at time $\bar{t} > 0$, either by a smooth or sudden transfer from $S \cup \hat{V}$ (including $\varpi = 0$). Here $\mathscr{M}(\bar{t}) \leqslant 0$ by earlier arguments ((i), (ii) and preceding remarks for $\varpi = 0$). It then follows from (4.8) that $\mathscr{M}$ remains non-positive for both cases (*a*) and (*b*), while N_0 remains in V.

In all cases $\mathscr{M} \leqslant 0$ and thus $\chi \leqslant u$. Finally, since the preceding arguments apply equally well to $-\chi$ in place of χ,

$$|\chi| \leqslant u, \quad t \geqslant 0. \tag{4.9}$$

Remarks

R.1. In cases (i) and (ii) of the proof it was necessary to omit the z-axis, since one of the coefficients in $\mathscr{E}$ is unbounded as $\varpi \to 0$. The theorems quoted from P.W. then apply to suitably restricted regions about N_0 (P.W., remark (i), p. 64). The unboundedness of $\hat{V}$ also causes no difficulty (P.W., remark (ii), p. 64).

R.2. The comparison theorem could alternatively be proven by considering the space–time maximum of $\pm\chi - u$ rather than just the space maximum. Since the space–time maximum is a space maximum, most of the proof remains unchanged. However, argument (iii) dealing with N_0 in V can then be replaced by an application of a maximum principle for parabolic differential inequalities (P.W., theorem 5, p. 173). Such an approach is more general since the presumption of piecewise differentiability of $\mathscr{M}(t)$ used in the present proof can be omitted. This alternative style of proof is illustrated in detail in §5.2 for A.

Extensions

The comparison theorem can be extended in several useful ways, which we note here for future reference.

E.1. If u_1 and u_2 satisfy (4.1), (4.2), (4.4), and if $u_1 \geqslant \chi$ and $u_2 \geqslant -\chi$ at $t = 0$, then

$$-u_2 \leqslant \chi \leqslant u_1, \quad t \geqslant 0. \tag{4.10}$$

E.2. Suppose $\mathscr{M}(t)$ does not occur at $\varpi = 0$ or $r = \infty$ for any t. Then, as argued earlier, N_0 must be in V and inequality (4.8) holds. Suppose that the space maximum $\mathscr{M}(t)$ remains constant during the time interval $t_1 \leqslant t \leqslant t_2$. Then $\mathscr{M}$ is the space–time maximum of $\chi - u$ in this time interval and it is attained by $\chi - u$ at an interior point of V. By a maximum principle for parabolic differential inequalities (P.W., theorem 5, p. 173) applied to (4.7) this is not possible unless $\chi - u \equiv \mathscr{M}$ in $[t_1, t_2]$ throughout V (V is connected), and by continuity and case (ii) of the proof for the comparison theorem, throughout V_∞. Thus if $\chi - u \not\equiv \mathscr{M}$ at any time t, $\partial(\chi - u)/\partial t$ can be zero on N_0 no more than instantaneously. So $\mathscr{M}$ must be strictly decreasing. In particular, $\mathscr{M}(t) < \mathscr{M}(0)$ for $t > 0$, and correspondingly, since $\mathscr{M}(0) \leqslant 0$, inequalities (4.9) and (4.10) can be strengthened to

$$|\chi| < u, \quad t > 0; \tag{4.11a}$$

$$-u_1 < \chi < u_2, \quad t > 0. \tag{4.11b}$$

E.3. Continuity of $\partial(\chi - u)/\partial n$ across S is used only to argue that N_0 cannot lie on S. The argument actually only requires $\partial(\chi - u)/\partial n < 0$ on N_0, if N_0 is on S. The argument would validly proceed if, for examples, $\partial(\chi - u)/\partial n < 0$ everywhere on (both sides of) S; or merely if sign $\{\partial(\chi - u)/\partial n\}$ was continuous at N_0, not necessarily negative everywhere on S, but, by the outward derivative theorem already quoted in (ii), definitely negative at N_0 when N_0 is on S. Thus $\partial u/\partial n$ or $\partial\chi/\partial n$, or both, may be discontinuous across S. This allows a wider choice of comparison functions, and is also used later in §6, G.1 to allow the possibility of a discontinuity in the permeability across S. Similar extensions hold for u_1, u_2 in E.1.

E.4. It is not necessary that $\partial^2 u/\partial r^2$ be continuous across $r = 1$. The proof of the comparison theorem is readily modified by dividing $\hat{V}$ into two regions, $\hat{V}_1$ where $r < 1$ and $\hat{V}_2$ where $r > 1$. Provided sign $\{\partial(\chi - u)/\partial n\}$ is continuous across the surface S and on $r = 1$, N_0 must lie in V and the comparison theorem remains valid. Similarly, $\partial^2 u/\partial \varpi^2$ may be discontinuous across $\varpi = 1$ provided sign $\{\partial(\chi - u)/\partial n\}$ is continuous across S and on $\varpi = 1$.

4.2. *Comparison functions for* χ: *decay to zero*

The usefulness of the comparison theorem is that it replaces the difficult problem of solving *equations* (2.16), (2.17) and (2.19) for χ by the comparatively simple problem of solving *inequalities* (4.1)–(4.4). In this section we will show how to systematically construct comparison functions u that decay to zero with increasing time. The comparison theorem then implies that χ must also decay to zero, and asymptotically at least as fast as u. However, it will be seen that these comparison functions decay extremely slowly for large R. We also give some simple comparison functions that reproduce and strengthen the results of previous authors, but which do not prove decay to zero nor indicate a decay rate.

4.2.1. *An axisymmetric comparison function that decays to zero*

Consider

$$u = \begin{cases} Y(0)\,F(\varpi)\,\mathrm{e}^{-pt}, & \varpi \leqslant 1, \qquad (4.12a) \\ Y(0)\,F(1)\,\mathrm{e}^{-pt}, & \varpi \geqslant 1, \qquad (4.12b) \end{cases}$$

$Y(0) > 0$, $F(\varpi) \geqslant 0$, $p > 0$, to be determined. We will rely on extension E.4 of the comparison theorem of §4.1, and not require $\partial^2 u/\partial\varpi^2$ to be necessarily continuous at $\varpi = 1$. Consider the conditions of the comparison theorem in turn.

Condition (4.1) is satisfied in $\hat{V}$, where $\varpi > 1$, since $\mathscr{E}u = 0$ there, and in $\hat{V}$, where $\varpi < 1$, if

$$\frac{\mathrm{d}^2F}{\mathrm{d}\varpi^2} - \frac{1}{\varpi}\frac{\mathrm{d}F}{\mathrm{d}\varpi} \leqslant 0. \qquad (4.13)$$

Also,

$$\mathscr{P}u = \mathrm{e}^{-pt}\,Y(0)\left\{\eta\left(\frac{\mathrm{d}^2F}{\mathrm{d}\varpi^2} - \frac{1}{\varpi}\frac{\mathrm{d}F}{\mathrm{d}\varpi} + \kappa\frac{\mathrm{d}F}{\mathrm{d}\varpi} + \lambda\right) - (\eta\kappa + Rv_\varpi)\frac{\mathrm{d}F}{\mathrm{d}\varpi} + (pF - \eta\lambda)\right\},$$

where λ, κ are yet to be specified. Thus condition (4.2) is satisfied if we choose F, κ, p such that for $\varpi \leqslant 1$

$$\frac{\mathrm{d}F}{\mathrm{d}\varpi} \geqslant 0, \quad \frac{\mathrm{d}^2F}{\mathrm{d}\varpi^2} - \frac{1}{\varpi}\frac{\mathrm{d}F}{\mathrm{d}\varpi} + \kappa\frac{\mathrm{d}F}{\mathrm{d}\varpi} = -\lambda, \qquad (4.14a, b)$$

$$\kappa = \sup_{V,\,t\geqslant 0}\left\{-\frac{Rv_\varpi}{\eta}, 0\right\}, \quad p \leqslant \inf_{V,\,t\geqslant 0}\left\{\frac{\eta\lambda}{F}\right\}. \qquad (4.14c, d)$$

The zero in (4.14c) is to ensure that $\kappa \geqslant 0$ even if $v_\varpi > 0$ always. This latter posibility can only occur if V never intersects the z-axis and S is always moving. Apart from this extremely remote but conceivable possibility, κ may simply be thought of as a magnetic Reynolds number based on v_ϖ. The non-negativity of κ will be needed later. Given (4.14a), conditions (4.3), (4.4) are satisfied if we choose $Y(0) = \max\{|\chi|/F(\varpi)\}$ at $t = 0$ and

$$F(0) > 0. \qquad (4.15)$$

($F(0) = 0$ is also a possibility, as will be seen later.)

For decay, p must be positive. And since F is positive by (4.14a) and (4.15), equation (4.14d) requires $\lambda > 0$ for decay. For simplicity, let λ be a positive constant. (Many other choices of λ have been explored but to no significant advantage.) Then (4.13) is satisfied if (4.14a, b) are, since $\kappa \geqslant 0$. A solution to (4.14a, b), (4.15), chosen with $\mathrm{d}F/\mathrm{d}\varpi = 0$ at $\varpi = 1$, is

$$F(\varpi) = F(0) + \lambda F_1(\varpi), \qquad (4.16)$$

where for $\varpi \leqslant 1$,

$$\left.\begin{aligned} F_1(\varpi) &= \int_0^{\varpi}\int_{\rho}^{1} \frac{\rho}{\eta}\,\mathrm{e}^{\kappa(\eta-\rho)}\,\mathrm{d}\eta\,\mathrm{d}\rho \\ &= \frac{1}{\kappa^2}[(1+\kappa\varpi)\,\mathrm{e}^{-\kappa\varpi}\{\mathrm{Ei}(\kappa\varpi)-\mathrm{Ei}(\kappa)\}+\mathrm{Ei}(\kappa)-\ln(\kappa\varpi)-\kappa\varpi-\gamma]. \end{aligned}\right\} \tag{4.17}$$

Equation (4.16) holds for all ϖ if we define $F_1(\varpi) = F_1(1)$, $\varpi \geqslant 1$.

In (4.17 *b*),

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{\mathrm{e}^x}{x}\,\mathrm{d}x$$

is an exponential integral (Abramowitz & Stegun 1972) and $\gamma = 0.577\ldots$ is the Euler constant.

Since $\max F = F(1)$, the maximum p satisfying (4.14 *d*) for a given λ, and unknown functional form of η, is $p = \lambda/F(1) > 0$. It follows from the comparison theorem that

$$|\chi| \leqslant Y(0)\,F(\varpi)\,\mathrm{e}^{-\lambda t/F(1)} \leqslant \frac{X(0)}{F(0)}F(1)\,\mathrm{e}^{-\lambda t/F(1)}. \tag{4.18a, b}$$

Increasing λ clearly increases the decay rate, but in (4.18 *b*) also increases the amplitude of the bounding function. An 'optimum' bounding function may be found by taking the envelope with respect to λ.

Let

$$\tau = \int_0^1\int_{\rho}^{1} \frac{\rho}{\eta}\,\mathrm{e}^{\kappa(\eta-\rho)}\,\mathrm{d}\eta\,\mathrm{d}\rho = \kappa^{-2}\{\mathrm{Ei}(\kappa)-\ln\kappa-\kappa-\gamma\},$$

so that $F(1) = F(0)+\lambda\tau$. Bearing in mind that $\lambda \geqslant 0$, the envelope of (4.18 *b*) with respect to λ, is $X(0)\,E(t;\tau)$ where

$$E(t;\tau) = \begin{cases} 1, & t \leqslant \tau, \\ \dfrac{t}{\tau}\mathrm{e}^{1-t/\tau}, & t \geqslant \tau. \end{cases} \tag{4.19}$$

The initial decay time of this envelope (i.e. the e-folding time) is approximately 3τ, whereas, for $t \gg \tau$, the decay time is approximately τ. By advancing the time origin, it follows that $X(t)$ decays monotonically to zero irrespective of compressibility or variable conductivity, and the decay time is bounded approximately by τ. Correspondingly, $|\chi|$ decays to zero, but not necessarily monotonically, and everywhere

$$|\chi| \leqslant X(t) \leqslant X(0)\,E(t;\tau). \tag{4.20}$$

From (4.18 *a*) it follows that the function

$$Y(t) = \max_{V_\infty}\{|\chi|/F(\varpi)\}$$

satisfies

$$Y(t) \leqslant Y(0)\,\mathrm{e}^{-\lambda t/F(1)};$$

and by advancing the time origin, we conclude $Y(t)$ must decay strictly monotonically to zero for all $\lambda > 0$. In particular, letting $\lambda \to \infty$ shows that

$$Y_1(t) \leqslant Y_1(0)\,\mathrm{e}^{-t/\tau}, \tag{4.21a}$$

where

$$Y_1(t) = \max_{V_\infty}\{|\chi|/F_1(\varpi)\} \tag{4.21b}$$

and $Y_1(t)$ decays strictly monotonically to zero. The same result (4.21) arises if the limiting case $F(0) = 0$ is considered in place of (4.15). But some care is needed since $F_1(0) = 0$. Obviously (4.21 *a*, *b*) are only meaningful if $F_1(\varpi)$ does not increase faster than χ, that is faster than $O(\varpi^2)$, for small ϖ (see (2.16 *b*)). However, $\exp\{\kappa(\eta-\rho)\} \geqslant 1$ since $\kappa \geqslant 0$, $\rho \leqslant \eta$. So (4.17 *a*) implies $F_1(\varpi) \geqslant \varpi^2(1-2\ln\varpi)/4 \geqslant \frac{1}{4}\varpi^2$, and thus $Y_1(t)$ defined by (4.21 *b*) is finite (taking, of course, the appropriate limit as $\varpi \to 0$). In terms of χ, (4.21 *a*, *b*) imply

$$|\chi| \leqslant Y_1(0)\, F_1(\varpi)\, \mathrm{e}^{-t/\tau} \leqslant Y_1(0)\, \tau \mathrm{e}^{-t/\tau}, \qquad (4.22\,a, b)$$

which, for $t > \max\{\tau, \tau^2 Y_1(0)/eX(0)\}$, is a tighter bound than (4.20). Since $F_1(\varpi) = O(\varpi)$ for small ϖ, (4.22 *a*) is also tighter than (4.20) for sufficiently small ϖ and all $t \geqslant 0$. Grouping together our uniform bounds, we conclude

$$|\chi| \leqslant \begin{cases} X(0), & t \leqslant \tau, \qquad (4.23\,a) \\ X(0)\,\dfrac{t}{\tau}\,\mathrm{e}^{1-t/\tau}, & t \geqslant \tau, \qquad (4.23\,b) \\ Y_1(0)\, F_1(\varpi)\, \mathrm{e}^{-t/\tau} & t \geqslant 0 \qquad (4.23\,c) \end{cases}$$

((4.23 *a*) will be strengthened to strict inequality for $t > 0$ in §4.2.3. This implies, by advancing the time origin, that $X(t)$ decays strictly monotonically.)

The uniform bounds in (4.23) can be immediately integrated, to prove, for example, that $\|\chi\|_2$ (see equation (3.3)) decays to zero, albeit not necessarily monotonically. It should be appreciated that uniform pointwise bounds like (4.23) generally have an advantage over integral results, in that for large t the former rule out the possibility of spiky behaviour, either in space or time, of χ. On the other hand, a decay result for $\|\chi\|_2$, for example, does not preclude spikes in χ. (If the spatial width of the spike $\to 0$ as $t \to \infty$, then so too may its contribution to $\|\chi\|_2$, regardless of the spike height.) Note, however, that (4.23) does not preclude spikes in the components of $\boldsymbol{B}_{\mathrm{m}}$, which need further analysis as in §4.3 and §4.4.

For given κ, τ may be evaluated with standard series expansions for Ei to obtain

$$\tau = \sum_{n=2}^{\infty} \frac{\kappa^{n-2}}{n n!} \qquad (4.24\,a)$$

$$= \frac{1}{\kappa^2}\left[\frac{e^{\kappa}}{\kappa}\left\{1+\frac{1!}{\kappa}+\frac{2!}{\kappa^2}+O\left(\frac{1}{\kappa^3}\right)\right\}-\ln\kappa-\kappa-\gamma\right], \quad \text{for large } \kappa. \qquad (4.24\,b)$$

Table 1 shows τ in the range $0 \leqslant \kappa \leqslant 100$.

TABLE 1. MAGNETIC REYNOLDS NUMBERS κ, κ_0 AND ASSOCIATED DECAY-TIME BOUNDS τ, τ_0 FOR THE MERIDIONAL FLUX FUNCTION χ

κ, κ_0	τ	τ_0
0	0.25	0.25
1	0.32	0.29
2	0.42	0.33
5	1.3	0.56
10	25	1.9
15	1×10^{3}	9.6
20	6×10^{4}	62
50	4×10^{16}	3×10^{7}
100	3×10^{37}	5×10^{17}

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In general v_ϖ takes both positive and negative values, so that $\kappa > 0$. For the exceptional case of an expanding star, where v_ϖ is everywhere non-negative, avoidance of mass sources requires $v_\varpi = 0$ at $\varpi = 0$, and correspondingly $\kappa = 0$. Similarly, for free decay, $\boldsymbol{v} \equiv \boldsymbol{0}$, and again $\kappa = 0$. For both of these cases (4.24*a*) gives $\tau = \frac{1}{4}$. (Recall from §2.2 that the time unit is a^2/η_0, where a is an upper bound for the radius of V and η_0 a lower bound for η.) Given that V is possibly time-dependent and otherwise quite general (see §2.1(i)), and that σ is variable, this value of τ is remarkably close to the theoretical (lowest poloidal mode) free-decay time $\tau_{\text{pol}} = \pi^{-2}$ for a sphere of uniform conductivity and unit radius. (Indeed, $\tau \approx 2.5\tau_{\text{pol}}$.)

On the other hand, for $\kappa \gg 1$, (4.24*b*) gives $\tau \sim \kappa^{-3}\mathrm{e}^{\kappa} \gg 1$. (For example, $\tau \simeq 3 \times 10^{37}$ when $\kappa = 100$, a value of κ possibly applicable to the Earth (Moffatt 1978).) This result leaves open the possibility that some axisymmetric dynamos may decay only extremely slowly on the diffusion timescale. While this slow decay rate bound may sometimes be accurate, it will on other occasions reflect an inadequacy in the chosen comparison function or the comparison theorem. An obvious example is that of incompressible flow and constant σ where, as mentioned in §3, the decay rate bounds of several integral parameters ($\|\chi\|_2$, M_1) of the field are close to the free-decay rate, independent of the magnitude of κ. For this special case, the bounding functions in (4.23) are necessarily slack since (4.23) is independent of the fluid dilatation rate $\nabla \cdot \boldsymbol{v}$. It would clearly be desirable to construct a comparison function that has a pointwise decay rate dependent not only on the 'magnetic Reynolds number' κ, but also on the fluid expansion rates. As yet no success has been attained in this regard. Some dramatic, but still not completely adequate, reduction in τ for large κ (i.e. large R) can be obtained for some cases by making additional assumptions on $\boldsymbol{v}$, as in the following example.

4.2.2. *An axisymmetric comparison function with v_ϖ/ϖ assumed uniformly bounded*

So far we have used only the continuity and boundedness of $\boldsymbol{v}$; but the axisymmetry and differentiability conditions (iii) and (v) of §2.1 imply v_ϖ/ϖ bounded for any finite t. In this section we will assume further that v_ϖ/ϖ is *uniformly* bounded for all t. We proceed in the same way as for comparison function (4.12), but now replace κ by $\kappa_0\varpi$, where κ_0 is a constant. Analogous to (4.14*b*, *c*), we choose

$$\frac{\mathrm{d}^2F}{\mathrm{d}\varpi^2} - \frac{1}{\varpi}\frac{\mathrm{d}F}{\mathrm{d}\varpi} + \kappa_0\varpi\frac{\mathrm{d}F}{\mathrm{d}\varpi} = -\lambda, \quad \text{and} \quad \kappa_0 = \sup_{v,\ t\geqslant 0}\left\{\frac{-Rv_\varpi}{\eta\varpi}, 0\right\}.$$

Analogous to (4.17), we find

$$F_1(\varpi) = \int_0^{\varpi}\int_0^1 \frac{\rho}{\eta}\,\mathrm{e}^{\frac{1}{2}\kappa_0(\eta^2-\rho^2)}\,\mathrm{d}\eta\,\mathrm{d}\rho \tag{4.25a}$$

$$= (2\kappa_0)^{-1}\left[\exp\left(-\tfrac{1}{2}\kappa_0\varpi^2\right)\left\{\mathrm{Ei}\left(\tfrac{1}{2}\kappa_0\varpi^2\right) - \mathrm{Ei}\left(\tfrac{1}{2}\kappa_0\right)\right\} + \mathrm{Ei}\left(\tfrac{1}{2}\kappa_0\right) - \ln\left(\tfrac{1}{2}\kappa_0\varpi^2\right) - \gamma\right]. \tag{4.25b}$$

Analogous to the uniform decay bounds (4.21) and (4.23), we obtain

$$|\chi| \leqslant \begin{cases} X(0)\,E(t;\tau_0), & (4.26a)\\ Y_1(0)\,\tau_0\,\mathrm{e}^{-t/\tau_0}, & (4.26b)\end{cases}$$

$$Y_1(t) \leqslant Y_1(0)\,\mathrm{e}^{-t/\tau_0}, \tag{4.27}$$

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where E and Y_1 are defined by (4.19) and (4.21 b), but with F_1 as in (4.25) and

$$\tau_0 = F_1(1) = (2\kappa_0)^{-1}\{\mathrm{Ei}\,(\tfrac{1}{2}\kappa_0) - \ln\,(\tfrac{1}{2}\kappa_0) - \gamma\} \tag{4.28a}$$

$$= \tfrac{1}{4}\sum_{n=2}^{\infty} \frac{(\kappa_0/2)^{n-2}}{(n-1)\,(n-1)\,!} \tag{4.28b}$$

$$= \frac{1}{2\kappa_0}\left[\frac{2}{\kappa_0}\,\mathrm{e}^{\frac{1}{2}\kappa_0}\left\{1 + \frac{2}{\kappa_0} + \frac{2^2 2!}{\kappa_0^2} + O\left(\frac{1}{\kappa_0^3}\right)\right\} - \ln\,(\tfrac{1}{2}\kappa_0) - \gamma\right], \quad \text{for large } \kappa_0. \tag{4.28c}$$

For $\kappa_0 = 0$, $\tau_0 = \frac{1}{4} \approx 2.5\,\tau_{\text{pol}}$, the same value as τ when $\kappa = 0$; and for small non-zero κ_0 (4.28 b) gives reasonable decay time bounds; see table 1. Whilst generally $\kappa_0 \geqslant \kappa$ and a direct comparison of τ and τ_0 is not possible, table 1 shows that in those cases where $\kappa \approx \kappa_0$ then $\tau_0 < \tau$. Indeed, for large κ_0 (4.28 c) gives much lower decay time bounds than τ in (4.24 b). The bounds are, however, still very large – $\tau_0 \approx 5 \times 10^{17}$ when $\kappa_0 = 100$ – and undoubtedly slack bounds for incompressible flow and other special cases.

4.2.3. *Other comparison functions and associated results*

The choice $u = X(0)$ satisfies conditions (4.1), (4.2), (4.3) and (4.4), and consequently, by the comparison theorem of §4.1, leads to

$$|\chi| \leqslant X(0). \tag{4.29}$$

By continuously advancing the time origin, it follows that $X(t)$ can never increase; the result obtained by the n.l. argument in §3. This can be strengthened according to extension E.2 of §4.1. Either (i) $\chi \equiv \pm X(t_0)$ at some time t_0 or (ii) $|\chi| < X(0)$ by (4.11). Since $\chi = 0$ at $\varpi = 0$ and at $r = \infty$, case (i) is only possible if $\chi \equiv 0$ at t_0, in which case (4.29), with t_0 chosen as time origin, implies $\chi \equiv 0$ for all $t \geqslant t_0$. By continuously advancing the time origin, case (ii) shows that $X(t)$ must strictly decrease, although not necessarily to zero, for any non-trivial χ. As mentioned in §3, this result has been obtained independently by a non-comparison function method by Lortz & Meyer-Spasche (1982 a). This proves the assertion made in §3, that the current density j_ϕ around the neutral line corresponding to $X(t)$ cannot be zero during any time interval. Note that in steady conditions only case (i) is possible and hence the only steady solution is $\chi \equiv 0$. This proves the steady antidynamo theorem quoted by Lortz (1968), and extends it to allow $\boldsymbol{v}/\eta$ to be discontinuous across S. (The proof of Lortz (1968) relies on a theorem on generalized analytic functions in Vekua (1962), p. 154), which does not seem to directly apply to χ, due in part to the unbounded coefficient $1/\varpi$ in equation (6) of Lortz (1968); for details, see Ivers (1984). However, the a.d.t. used by Lortz may be proven either as shown in this section or merely by quoting modern versions of appropriate maximum principles for elliptic differential equations (P.W.; Gilbarg & Trudinger 1977).)

The choices $u_1 = \max \chi(t=0)$, $u_2 = -\min \chi(t=0)$ satisfy (4.1), (4.2), (4.4) and the requirements of extension E.1. Substitution into (4.10) gives the Backus non-amplification result

$$\min_{V_\infty} \chi(t=0) \leqslant \chi(t) \leqslant \max_{V_\infty} \chi(t=0), \quad t \geqslant 0.$$

Again, for non-trivial χ, this can be strengthened by extension E.2 to

$$\min_{V_\infty} \chi(t=0) < \chi(t) < \max_{V_\infty} \chi(t=0), \quad t > 0;$$

and advancing the time origin shows that $\max \chi(t)$ must strictly monotonically decrease, and $\min \chi(t)$ must strictly monotonically increase, but neither necessarily to zero without further argument.

Finally we mention that one can construct spherically symmetric comparison functions analagous to §§4.2 (i) and (ii), and determine functions $F(r), F_1(r)$ in place of $F(\varpi), F_1(\varpi)$. Indeed, when S is a sphere of unit radius, one can even incorporate the velocity boundary condition, $v_r = 0$ on S, by replacing κ by $\kappa_0(1-r)$ (on the further assumption that $v_r/(1-r)$ is bounded in space–time). These spherically symmetric comparison functions yield similar, but usually somewhat slacker, bounds than their z-independent counterparts.

4.3. *Decay to zero of the external multipole moments and induction vector $\boldsymbol{B}_{\mathrm{m}}$ in $\hat{V}$*

The current-free and solenoidal conditions (2.10) and (2.7c) imply the existence of Ψ such that $\boldsymbol{B} = -\nabla\Psi$, $\nabla^2\Psi = 0$, in $\hat{V}$. In that part of $\hat{V}$ where $r > 1$, Ψ is given by a spherical harmonic expansion that serves to define the external multipole moments $M_k(t)$:

$$\Psi = \sum_{k=1}^{\infty} \frac{M_k(t)}{r^{k+1}} P_k(\cos\theta), \tag{4.30}$$

where P_k is a Legendre polynomial. Alternatively, (2.19) can be solved for χ in $r > 1$:

$$\chi = \sum_{k=1}^{\infty} \frac{E_k(t)}{r^k} P_{k,1}(\cos\theta)\sin\theta, \tag{4.31}$$

where $P_{k,1}$ is the Neumann form of the associated Legendre function (Chapman & Bartels (1962), §17.2). M_k and E_k can then be related by

$$B_\theta = -\frac{1}{r}\frac{\partial\Psi}{\partial\theta} = -\frac{1}{r\sin\theta}\frac{\partial\chi}{\partial r}.$$

Using (4.30), (4.31) and the property

$$\mathrm{d}P_k(\cos\theta)/\mathrm{d}\theta = -P_{k,1}(\cos\theta) \tag{4.32}$$

one finds

$$E_k(t) = M_k(t)/k.$$

Substitution of this into (4.31), and use of (4.32) and the orthogonality property

$$\int_0^\pi P_{k,1}(\cos\theta)\, P_{l,1}(\cos\theta)\sin\theta\,\mathrm{d}\theta = \frac{(k+1)!}{(k-1)!}\frac{2\delta_{kl}}{2k+1}$$

gives

$$M_k(t) = \frac{2k+1}{2k}\int_0^\pi \chi(1,\theta,t)\, P_{k,1}\,\mathrm{d}\theta. \tag{4.33}$$

Now consider the Legendre expansion

$$P_k(\cos\theta) = \frac{1\times 3\times\ldots(2k-1)}{2\times 4\times\ldots(2k)}\left\{2\cos k\theta + \frac{1\times 2k}{2\times(2k-1)} 2\cos(k-2)\,\theta + \ldots\right\}. \tag{4.34}$$

Differentiation of (4.34) and use of (4.32) yields the bound

$$|P_{k,1}| \leqslant k\left\{\frac{1\times 3\times\ldots(2k-1)}{2\times 4\times\ldots(2k)}\left(2 + \frac{1\times 2k}{2\times(2k-1)}2 + \ldots\right)\right\}, \tag{4.35a}$$

$$= kP_k(1) = k. \tag{4.35b}$$

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So, (4.33) implies

$$|M_k(t)| \leqslant (k+\tfrac{1}{2})\,\pi\, X(t). \tag{4.36}$$

Thus the multipole moments $M_k(t)$ must decay to zero no slower than $X(t)$ and no slower than the bounds in (4.23). Equation (4.36) does not imply monotonic decay of $|M_k(t)|$, but for large t does rule out timewise spikes such as those contemplated by Backus (1957, figure 1) in the dipole moment M_1.

In the region $r \geqslant \xi$ for any $\xi > 1$, the expansion (4.30) for Ψ can be differentiated term by term (since the resulting series are uniformly convergent). Thus

$$B_r = \sum_{k=1}^{\infty} \frac{(k+1)\,M_k(t)}{r^{k+2}} P_k(\cos\theta), \quad B_\theta = \sum_{k=1}^{\infty} \frac{M_k(t)}{r^{k+2}} P_{k,1}(\cos\theta).$$

Consequently, since $|P_k(\cos\theta)| \leqslant 1$,

$$|B_r| \leqslant \sum_{k=1}^{\infty} \frac{(k+1)\,|M_k(t)|}{r^{k+2}};$$

or, by (4.36)

$$|B_r| \leqslant \sum_{k=1}^{\infty} \frac{(k+1)\,(k+\frac{1}{2})\,\pi}{r^{k+2}} X(t) = \frac{(6r^2-3r+1)\,\pi}{2r^2(r-1)^3} X(t). \tag{4.37 a, b}$$

Similarly, by using the bound (4.35),

$$|B_\theta| \leqslant \frac{(6r^2-3r+1)\,\pi}{2r^2(r-1)^3} X(t). \tag{4.38}$$

Finally, from (4.37) and (4.38),

$$|\boldsymbol{B}_{\mathrm{m}}| \leqslant \frac{(6r^2-3r+1)\,\pi}{\sqrt{2}\, r^2(r-1)^3} X(t), \tag{4.39}$$

which establishes the decay to zero of $\boldsymbol{B}_{\mathrm{m}}$ in $r > 1$, at a rate no slower than $X(t)$. Note that the bound (4.39) is non-uniform in $r > 1$, owing to the term $(r-1)^3$ in the denominator and that (4.39) holds only in that region of $\hat{V}$ where $r > 1$. An alternative bound that holds throughout all $\hat{V}$ can be derived by using an *a priori* interior estimate for elliptic differential equations. The details are notationally complicated and therefore deferred to Appendix B, part (*a*), where it is shown

$$|\boldsymbol{B}_{\mathrm{m}}| \leqslant CX(t)/\varpi d \quad \text{in } \hat{V}, \tag{4.40}$$

where C is a constant and d is the shortest distance from the field point to $S \cup \{\varpi = 0\}$ at time t. Equation (4.40) shows that $\boldsymbol{B}_{\mathrm{m}}$ decays no slower than X in $\hat{V}$, but, as in (4.39), the amplitude of the bound in (4.40) is again non-uniform, being unbounded on S and the z-axis. This behaviour on the z-axis is due to the unbounded coefficient in the operator $\mathscr{E}$.

4.4. *Decay to zero of* $\boldsymbol{B}_{\mathrm{m}}$ *in* V *and of the current density* j_ϕ

In an analogous way to (4.40), an *a priori* interior estimate for parabolic partial differential equations applied to (2.17) provides non-uniform pointwise decay bounds on $\boldsymbol{B}_{\mathrm{m}}$ and the associated azimuthal electric current density j_ϕ. As shown in Appendix B, part (*b*),

$$|\boldsymbol{B}_{\mathrm{m}}| \leqslant \frac{K}{\varpi d} X(t-\tfrac{1}{4}), \quad t \geqslant \tfrac{1}{4}, \tag{4.41}$$

$$|\mu_0 j_\phi| \leqslant K\left(\frac{1+2d}{\varpi d^2}\right) X(t-\tfrac{1}{4}), \quad t \geqslant \tfrac{1}{4}, \tag{4.42}$$

where K is a constant and d is the shortest distance from the field point to $S \cup \{\varpi = 0\}$ during the interval $(t-\frac{1}{4}, t)$. Since $X(t-\frac{1}{4})$ decays to zero by (4.23), it follows that both $\boldsymbol{B}_{\mathrm{m}}$ and j_ϕ decay to zero in $V-\{\varpi = 0\}$. Together, (4.40), (4.41) and (4.42) preclude the possibility that any spikes in $\boldsymbol{B}_{\mathrm{m}}$ or j_ϕ, such as contemplated in §3, might persist for large t inside $V \cup \hat{V}-\{\varpi = 0\}$. However, the non-uniformity of the bounds (4.40), (4.41), (4.42), caused by the presence of ϖ and d in the denominators, prevents us from concluding decay to zero of j_ϕ on S, or of $\boldsymbol{B}_{\mathrm{m}}$ on S and the z-axis (of course, $j_\phi \equiv 0$ at $\varpi = 0$). We conjecture that this is a technical difficulty awaiting appropriate mathematical resolution, rather than being any manifestation of ultimate spiky behaviour on S or $\{\varpi = 0\}$.

4.5. *Decay to zero of the net absolute surface flux*

A quantity of considerable interest because of its use in testing whether the geomagnetic secular variation can be regarded as the product of core-surface motions in a 'frozen-in' field approximation (Backus (1968); Booker (1969)) and in the associated estimation of planetary core radii from magnetic observations (Hide (1978); Hide & Malin (1979)) is the net absolute surface flux

$$\mathscr{F} = \int_S |\boldsymbol{B}\cdot \boldsymbol{dS}|.$$

By (2.15),

$$\mathscr{F} = 2\pi \int_0^l \left|\frac{\partial \chi}{\partial s}\right| \mathrm{d}s, \tag{4.43}$$

where s is the meridional arc length along S measured from the highest point, and l is the length of the intersection of S and any meridional half-plane $\{\phi = \text{constant}\}$.

Equation (4.43) shows that $\mathscr{F}$ decays to zero if and only if $\partial\chi/\partial s$ decays to zero almost everywhere. But decay of $\mathscr{F}$ does not follow from decay of χ alone, unless further assumptions are made.

One probably reasonable assumption is that the number, $N(t)$ say, of null-flux curves (where $B_n = 0$) on S is finite for finite t. (For the Earth, N is about three at present (Booker 1969).) Then, by using the flux interpretation of χ it is simple to show that

$$\mathscr{F} \leqslant 4\pi N(t)\, X(t).$$

So, provided N does not increase as fast as $\{X(t)\}^{-1}$, $\mathscr{F}$ will decay to zero. If N is bounded, $\mathscr{F}$ will ultimately decay at least as fast as $X(t)$, and at least as fast as indicated by the uniform bounds in (4.23).

One can use (4.37) to bound the net flux through spheres enclosing S, one particular application being to spherical core–mantle systems. Here, $r = 1$ would correspond to the boundary of the conducting core, and by (4.37) the net absolute flux through the non-conducting mantle's surface, $r = b > 1$, satisfies

$$\mathscr{F} \leqslant \frac{2\pi^2(6b^2-3b+1)\,X(t)}{(b-1)^3} \to 0 \quad \text{as} \quad t \to \infty.$$

(For a mantle–core system the size of the Earth's, $b = 1.84$ and the preceding bound reduces to $\mathscr{F} \lesssim 530\, X(t)$.)

5. Decay of the azimuthal field

5.1. *Prologue*

We will first consider decay of A similarly to our treatment of χ. However, a new comparison theorem is needed since $A \equiv 0$ in $\hat{V}$ and on S, and since the induction equation (2.23) for A contains source terms as discussed in §3. Indeed, it will not be possible to treat the source term $(v_\phi \boldsymbol{B}_\mathrm{m} A)$ precisely. But rather than neglect it entirely we will, as a first approximation, model it by an uniformly decaying function. We emphasize this as an assumption since although §4.3 and §4.4 showed that $\boldsymbol{B}_\mathrm{m} \to 0$ as $t \to \infty$, it has never been proven that $\boldsymbol{B}_\mathrm{m} \to 0$ uniformly in space. Specifically, we hereinafter assume the uniform bound

$$|R\boldsymbol{B}_\mathrm{m} \cdot \nabla (v_\phi/\varpi)| \leqslant \alpha(0)\, C(t), \tag{5.1}$$

where
$$0 \leqslant C(t) < \infty, \quad C(t) \to 0 \quad \text{as} \quad t \to \infty, \tag{5.2}$$

and $\alpha(0) > 0$, is a constant that we specify later. Introducing $\mathscr{A}(t) = \max |A|$ at time t, we assume $\mathscr{A}(0) \neq 0$. If $\mathscr{A}(0) = 0$, then either A remains zero if $(v_\phi \boldsymbol{B}_\mathrm{m} A) = 0$, or non-zero A is generated by differential rotation acting on $\boldsymbol{B}_\mathrm{m}$. For the latter we simply advance the time origin conveniently so that $\mathscr{A}(0) \neq 0$.

Since a new comparison theorem is needed, we will take the opportunity of illustrating the alternative parabolic maximum principle proof mentioned in remark R.2 of §4.1.

5.2. *Comparison theorem for the azimuthal field $A = B_\phi/\varpi$*

Theorem. *Suppose w is continuous in V and has continuous first and second derivatives in V, excluding the z-axis. If, with $\mathscr{Q}$ and c as in* (2.23)–(2.25),

$$(\mathscr{Q}+c)\, w \leqslant -\alpha(0)\, C(t) \quad \textit{in } V(t), \quad t \geqslant 0, \tag{5.3}$$

$$w \geqslant |A| \quad \textit{in } V, \quad \textit{at } t = 0, \tag{5.4}$$

$$\partial w/\partial \varpi \leqslant 0 \quad \textit{as } \varpi \to 0, \tag{5.5}$$

$$w \geqslant 0 \quad \textit{on } S, \tag{5.6}$$

then
$$|A| \leqslant w \quad \textit{in } V(t), \quad t \geqslant 0.$$

Proof. Let c_s be an upper bound on c, as in (2.28a), and let

$$Z = (A - w)\, \mathrm{e}^{-c_\mathrm{s} t}.$$

Then equation (2.23) and inequalities (5.1) and (5.3) imply

$$\{\mathscr{Q} + (c - c_\mathrm{s})\}\, Z \geqslant 0. \tag{5.7}$$

We proceed to show that $Z \leqslant 0$ in $V(t)$, $t \geqslant 0$.

Since Z is continuous, its space–time maximum $M(T)$ on the compact set $V \cup S(t)$, $0 \leqslant t \leqslant T$ exists. We assume that $M(T)$ is positive for some $T > 0$, otherwise there is nothing to prove. Clearly, $M(T)$ cannot occur in $V(0)$ by (5.4) or on S by (2.27) and (5.6). On the z-axis, $\partial Z/\partial \varpi \geqslant 0$ by (2.22) and (5.5). Application of an appropriate boundary derivative maximum principle for parabolic differential inequalities (see remark following) shows that if $M(T)$ occurs on the z-axis, then $M(t)$ also occurs at an interior point of V at time $\bar{t}$, where $0 < \bar{t} \leqslant T$. Finally, if $M(T)$ occurs in $V(\bar{t})$ then it follows from an appropriate maximum principle for parabolic differential inequalities that $Z \equiv M(T)$ in $V(t)$, for all t where $0 \leqslant t \leqslant \bar{t}$. Hence, by continuity, $Z \equiv M(T)$ in $V(0)$ and on $S(t)$ for $0 \leqslant t \leqslant \bar{t}$; and this, according to (5.4), (2.27) and (5.6), contradicts our assumption that $M(T) > 0$. Thus $M(T)$ cannot be positive for any T, and hence $w \geqslant A$ in $V(t)$ for all $t \geqslant 0$. A similar argument shows that $-w \leqslant A$ in $V(t)$ for all $t \geqslant 0$, which completes the proof of the theorem.

Remark. The 'appropriate' boundary derivative and maximum principle theorems required by the above proof stem from theorems 5, 6, 7 and associated remarks of P.W. (pp. 173–175). These P.W. theorems do not apply directly to the case considered here, owing to the occurrence of an unbounded coefficient $3\eta/\varpi$ in $\mathscr{Q}$ (see equation (2.24)). However, as noted elsewhere by P.W. (see their remark (iv), p. 170), results such as theorem 5 are still valid when the coefficients in the differential operator are merely bounded in all closed subsets of $V - \{\varpi = 0\}$. This is certainly true of $3\eta/\varpi$. That the conclusion of theorem 6 of P.W. remains valid is rather more fortuitous, being dependent on special features of the problem here, namely axisymmetry and the fact that $\eta > 0$. These features ensure that the sign of the unbounded term disturbing the proof of P.W. (p. 171) is such as to preserve the conclusions of the proof. (An alternative way to circumvent the unboundedness of $3\eta/\varpi$ is to write $\mathscr{Q}$ in terms of a five-dimensional Laplacian (Backus & Chandrasekhar 1956).) Theorem 7 of P.W. is a straightforward extension of theorems 5 and 6, following from the fact that $c - c_s$ is bounded and non-positive: $c_i - c_s \leqslant c - c_s \leqslant 0$ by (2.28).

5.3. *Comparison function for A: conditional decay to zero*

Proceeding as far as possible analogously to §4.2, let us consider

$$w = \alpha(0)\, G(\varpi)\, H(t). \tag{5.8}$$

Since

$$(\mathscr{Q}+c)\, w = \alpha(0) \left\{ \eta \left(\frac{d^2G}{d\varpi^2} + \frac{3}{\varpi}\frac{dG}{d\varpi} - \kappa \frac{dG}{d\varpi} + \lambda \right) H \right.$$
$$\left. + \left(\eta\kappa - Rv_\varpi + \frac{\partial \eta}{\partial \varpi} \right) H \frac{dG}{d\varpi} + (cG - \eta\lambda)\, H - G\frac{dH}{dt} \right\},$$

where κ, λ are yet to be specified, inequalities (5.3), (5.5) are satisfied if we choose

$$\frac{d^2G}{d\varpi^2} + \frac{3}{\varpi}\frac{dG}{d\varpi} - \kappa\frac{dG}{d\varpi} = -\lambda, \quad \kappa = \sup_{V,\, t\geqslant 0} \left\{ \frac{Rv_\varpi}{\eta} - \frac{\partial(\ln \eta)}{\partial \varpi}, 0 \right\}, \tag{5.9a, b}$$

$$dG/d\varpi \leqslant 0, \quad H \geqslant 0, \tag{5.10a, b}$$

and

$$-G\, dH/dt + (cG - \eta\lambda)\, H \leqslant -C(t). \tag{5.11}$$

The zero in (5.9b) is to ensure $\kappa \geqslant 0$, which will be used later. But this zero is unnecessary if V intersects the z-axis (where $v_\varpi = \partial\eta/\partial\varpi = 0$) at any time t. Furthermore, if we also choose

$$H(0) = 1, \quad G(1) > 0, \quad \alpha(0) = \max_{t=0} \{|A|/G(\varpi)\}, \tag{5.12a, b, c}$$

conditions (5.4), (5.6) are satisfied.

Analogous to (4.16), (4.17), a solution to (5.9*a*), (5.10*a*), (5.12*b*), obtained by assuming $\lambda \geqslant 0$ and by choosing $\mathrm{d}G/\mathrm{d}\varpi = 0$ at $\varpi = 0$, is

$$G(\varpi) = G(1) + \lambda G_1(\varpi), \tag{5.13a}$$

where

$$G_1(\varpi) = \int_\varpi^1 \frac{\mathrm{e}^{\kappa\xi}}{\xi^3}\left(\int_0^\xi \rho^3\,\mathrm{e}^{-\kappa\rho}\,\mathrm{d}\rho\right)\mathrm{d}\xi. \tag{5.13b}$$

Since $\eta \geqslant 1$ by (2.11*e*), inequalities (5.10*b*) and (5.11) are satisfied if

$$\frac{\mathrm{d}H}{\mathrm{d}t} + \left(\frac{\lambda}{G(0)} - c_\mathrm{s}\right)H = \frac{C(t)}{G(1)} = C'(t), \quad \text{say,}$$

which, taking (5.12*a*) into account, implies

$$H = \mathrm{e}^{-qt}\left\{1 + \int_0^t C'(s)\,\mathrm{e}^{qs}\,\mathrm{d}s\right\}, \tag{5.14a}$$

where

$$q = \lambda/G(0) - c_\mathrm{s}. \tag{5.14b}$$

Thus, given (5.2), $H \to 0$ as $t \to \infty$ if $q > 0$, i.e. if we can find a non-negative λ such that

$$\lambda > G(0)\,c_\mathrm{s}. \tag{5.15}$$

Substitution from (5.13*a*) shows that (5.15) can be written as

$$\lambda > \frac{G(1)\,c_\mathrm{s}}{1 - c_\mathrm{s}\tau_1(\kappa)},$$

provided

$$c_\mathrm{s}\tau_1(\kappa) < 1, \tag{5.16}$$

where

$$\tau_1(\kappa) = G_1(0) = \int_0^1 \frac{\mathrm{e}^{\kappa\xi}}{\xi^3}\left(\int_0^\xi \rho^3\,\mathrm{e}^{-\kappa\rho}\,\mathrm{d}\rho\right)\mathrm{d}\xi. \tag{5.17}$$

If condition (5.16) is violated, then q is non-positive, and the most we can conclude is that $H(t)$ grows no faster than $\mathrm{e}^{|q|t}$. If condition (5.16) is satisfied, and if $C(t)$ decays with characteristic time τ_c, then $H(t)$ ultimately decays with a characteristic time $\max(q^{-1}, \tau_\mathrm{c})$.

In principle, one might take the envelope of the comparison function defined by (5.8), (5.13), (5.14), with respect to the parameter λ, but for general $C(t)$, no simple formula is obtained. Summarizing the application of the comparison theorem, we have

$$|A| \leqslant \alpha(0)\,G(\varpi)\,\mathrm{e}^{-qt}\left(1 + \int_0^t C'(s)\,\mathrm{e}^{qs}\,\mathrm{d}s\right), \tag{5.18}$$

and decay of A follows contingent on condition (5.16). This condition is most obviously satisfied (i) when the dilatation rate $\nabla\cdot\boldsymbol{v}$ and diffusivity gradient $\partial\eta/\partial\varpi$ are small, so that c_s is sufficiently small; and (ii) when $c_\mathrm{s} \leqslant 0$.

If $c_\mathrm{s} \leqslant 0$ (i.e. $2\,\partial\eta/\partial\varpi \leqslant \varpi R\nabla\cdot\boldsymbol{v}$; see (2.25)), then (5.16) is trivially satisfied, and any $\lambda > 0$ will give decay to zero of A. The most obvious application is to an incompressible uniform fluid where $c_\mathrm{s} = \nabla\cdot\boldsymbol{v} = \partial\eta/\partial\varpi = 0$. Then $q = \lambda/G(0) \geqslant 0$ and (5.9*b*) reduces to

$$\kappa = \sup\{Rv_\varpi/\eta, 0\}.$$

If, in addition, $(v_\phi \boldsymbol{B}_\mathrm{m} A) = 0$, then analogous to (4.18)

$$|A| \leqslant \alpha(0)\, G(\varpi)\, \mathrm{e}^{-\lambda t/G(0)} \leqslant \frac{\mathscr{A}(0)}{G(1)} G(0)\, \mathrm{e}^{-\lambda t/G(0)}. \tag{5.19a, b}$$

Letting $\lambda \to \infty$ in (5.19a), and also taking the envelope of (5.19b) with respect to $\lambda \geqslant 0$ yields, analogous to (4.22), (4.23), the bounds

$$|A| \leqslant \begin{cases} \alpha_1(0)\, G_1(\varpi)\, \mathrm{e}^{-t/\tau_1}, & (5.20a) \\ \alpha_1(0)\, \tau_1\, \mathrm{e}^{-t/\tau_1}, & (5.20b) \\ \mathscr{A}(0)\, E(t;\, \tau_1), & (5.20c) \end{cases}$$

where $\alpha_1(t) = \max\{|A|/G_1(\varpi)\}$ over $V(t)$. Here E is as in (4.19) and has decay time τ_1 given by (5.17). For $t > \max\{\tau_1, \tau_1^2\, \alpha_1(0)/e\mathscr{A}(0)\}$, (5.20$a, b$) are tighter bounds than (5.20c); and since $G_1(1) = 0$, (5.20a) is tighter than (5.20c) for ϖ sufficiently close to 1 and all $t \geqslant 0$. (In the same way as for (4.23a), it is similarly possible to extend (5.20c) to see that $\mathscr{A}(t)$ decays strictly monotonically.)

In the very special case of free decay of a uniform conducting fluid, $C(t) = R = \kappa = 0$, and from (5.17) $\tau_1 = \frac{1}{8}$, not far removed from the toroidal free-decay time for a sphere of constant diffusivity and unit radius, namely $\tau_\mathrm{tor} \approx (1\backslash 43\pi)^{-2} \approx \frac{1}{20}$. Thus, merely changing to non-spherical shape canot extend the free-decay time by more than a factor of about 2.5.

The preceding special case indicates that the comparison function defined by (5.8), (5.13) and (5.14) probably gives reasonably accurate decay bounds for sufficiently small R, fluid strain rates and variations in η. However, the method fails when $\kappa \gg 1$. For, as shown in Appendix A, $\tau_1(\kappa) \sim 6\mathrm{e}^\kappa/\kappa^5$; and, from (2.28$a$), c_s increases linearly with R. Thus, for large κ, and hence presumably large R, (5.16) will be greatly violated. Bearing in mind (§3) that for a uniform incompressible fluid sphere, and when $(v_\phi \boldsymbol{B}_\mathrm{m} A) = 0$, Backus (1957) has shown that $\mathscr{A}(t)$ decays to zero at no slower than the poloidal free-decay rate τ_pol^{-1}, regardless of the size of κ or R, it is clear that in certain cases the comparison function of (5.8), (5.13) and (5.14) is grossly inadequate for $\kappa \gg 1$. A comparison function depending on the fluid dilatation rate $\nabla \cdot \boldsymbol{v}$ as well as the 'Reynolds number' κ would be preferable, but no success has been had in this regard. Analogous to §4.2.2, some improvement (i.e. reduction in τ_1) for large κ can be achieved by making the further assumption that v_ϖ/ϖ is uniformly bounded, but we omit the details.

As in §4.4 and Appendix C, we can obtain Schauder-type *a priori* estimates for the space–time derivatives of A, and hence, in particular, derive a conditional decay result for the meridional current density $\boldsymbol{j}_\mathrm{m} = \mu_0^{-1} \nabla \times (A\varpi\, \boldsymbol{e}_\phi)$. Details are left as a non-trivial exercise for the more-mathematically inclined readers (or see Ivers 1984). Here we merely give the result

$$|\boldsymbol{j}_\mathrm{m}| \leqslant \frac{K'\varpi}{\mu_0 d} \{\max_{V(t-\frac{1}{4})} |A| + RX(t-\tfrac{1}{4})\} + \frac{2}{\mu_0} \max_{V(t)} |A|,$$

where d is the same as in (4.41) and K' is constant, and from which decay follows corresponding to the decay of A and χ.

5.4. *The rate of change of* $\|A\|_1$: *decay when* $(v_\phi \boldsymbol{B}_\mathrm{m} A) = 0$

To strengthen our decay result for A, it is necessary to take into account the special divergence nature of (2.26). An integral method is suggested.

At any time t, $V(t)$ is divisible into toroidal sub-volumes V_l ($l = 1, 2, \ldots$), in which A does not change sign. Let the surface of these volumes be $S_l(t)$. Let $I_l = V_l \cap \{\varpi = 0\}$ if it is not null; and let S_ϵ be an associated small cylindrical surface of radius ϵ, such that $S_\epsilon \to I_l$ as $\epsilon \to 0$. Note that

$$|A| \equiv 0 \quad \text{and} \quad \partial|A|/\partial n \leqslant 0, \quad \text{on } S_l, \tag{5.21 a, b}$$

(the outward normal derivative here being formed from the inside of S_l). From (2.26) and (5.21 a),

$$\int_{V_l} \frac{\partial|A|}{\partial t}\,\mathrm{d}V = \lim_{\epsilon\to 0}\int_{S_l+S_\epsilon}\left\{\frac{\eta}{\varpi^2}\nabla(\varpi^2|A|) - R\boldsymbol{v}\,|A|\right\}\cdot \boldsymbol{dS} - \operatorname{sign}(A)\int_{V_l}(v_\phi \boldsymbol{B}_{\mathrm{m}} A)\,\mathrm{d}V,$$

$$= \int_{S_l}\eta\frac{\partial|A|}{\partial n}\,\mathrm{d}S - 4\pi\int_{I_l}\eta\,|A|\,\mathrm{d}z - \operatorname{sign}(A)\int_{V_l}(v_\phi \boldsymbol{B}_{\mathrm{m}} A)\,\mathrm{d}V. \tag{5.22}$$

By using (2.27) and summing (5.22) over l we calculate the rate of change of $\|A\|_1$ while allowing for the motion of S:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V |A|\,\mathrm{d}V = \int_V \frac{\partial|A|}{\partial t}\,\mathrm{d}V + \int_S |A|\,\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}$$

$$= \sum_l \int_{S_l}\eta\frac{\partial|A|}{\partial n}\,\mathrm{d}S - 4\pi\int_I \eta\,|A|\,\mathrm{d}z - \int_V \operatorname{sign}(A)\cdot(v_\phi \boldsymbol{B}_{\mathrm{m}} A)\,\mathrm{d}V, \tag{5.23}$$

where $I = V \cap \{\varpi = 0\}$. (With such mechanisms as the Nernst–Ettingshausen effect in mind – see §6, G.6 – note that since $A = 0$ on S, nowhere does (5.23) rely on $\boldsymbol{v}$ being interpreted as the fluid velocity.)

The right side of (5.23) can be bounded above by using (5.21 b), (2.11 a) and (5.1):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V |A|\,\mathrm{d}V \leqslant -\int_V \operatorname{sign}(A)\cdot(v_\phi \boldsymbol{B}_{\mathrm{m}} A)\,\mathrm{d}V \tag{5.24}$$

$$\leqslant \tfrac{4}{3}\pi\alpha(0)\,C(t). \tag{5.25}$$

Thus

$$\int_V |A|\,\mathrm{d}V \leqslant \left(\int_V |A|\,\mathrm{d}V\right)_{t=0} + \tfrac{4}{3}\pi\alpha(0)\int_0^t C(t)\,\mathrm{d}t, \tag{5.26}$$

which shows that $\|A\|_1$ is bounded above by

$$\left(\int_V |A|\,\mathrm{d}V\right)_{t=0} + \tfrac{4}{3}\pi\alpha(0)\int_0^\infty C(t)\,\mathrm{d}t, \tag{5.27}$$

which is finite if $C(t)$ decays exponentially as expected from (4.41).

The preceding results can be strengthened in the absence of a meridional field or differential rotation, when $(v_\phi \boldsymbol{B}_{\mathrm{m}} A) = 0$. To do this we must prove the impossibility of a non-trivial toroidal field A_0, which, at any time $t_0 \geqslant 0$, satisfies both (i) $\partial|A_0|/\partial n \equiv 0$ on S_l, for all l; and (ii) $A_0 \equiv 0$ on I, if I is not null; thereby making the right side of (5.23) zero. That $A_0 \equiv 0$ for all space and $t \leqslant t_0$ follows from the same parabolic boundary derivative and maximum principles used in the comparison theorem of §4.1, but here applied to the function $Z = -|A_0| \exp(-c_{\mathrm{s}} t)$, which satisfies (5.7) (with equality) in V_l. The argument is quite simple: $Z \leqslant 0$ in V_l and therefore takes its maximum (zero) on $S_l \cup I_l$. By theorem 7 of P.W. (p. 174), it follows that either (iii) $\partial Z/\partial n > 0$ on $S_l \cup I_l$, or (iv) Z is constant for $t \leqslant t_0$. Conditions (i) and (iii) are contradictory. Thus the only possibility is the implication of (ii), (iv) and (5.21*a*), namely $Z \equiv A_0 \equiv 0$ for $t \leqslant t_0$. It further follows that $A_0 \equiv 0$ for all t by 'integrating' (2.23) under the trivial initial boundary conditions $A_0 \equiv 0$ at $t = t_0$ and on $S_l \cup I_l$ for all l, t. We observe then that the right side of (5.23) (with $(v_\phi \boldsymbol{B}_\mathrm{m} A) = 0$) must be strictly negative unless $A \equiv 0$. In steady conditions the left side of (5.23) vanishes, and thus the only possibility is $A \equiv 0$. This argument provides an alternative proof of the steady toroidal a.d.t. of Lortz (1968) with the advantage of not requiring an existence theorem for the adjoint equation as done by Lortz. In non-steady conditions the implication of (5.23) (with $(v_\phi \boldsymbol{B}_\mathrm{m} A) = 0$) is that any non-trivial $\|A\|_1$ must be strictly monotonically decreasing for all t. The obvious physical expectation in the presence of $(v_\phi \boldsymbol{B}_\mathrm{m} A)$, is that $\|A\|_1$ will grow to some bound less than (5.27). And thereafter, taking into account the decay to zero of $(v_\phi \boldsymbol{B}_\mathrm{m} A)$ ensured by (4.41), $\|A\|_1$ will decrease. Note that we have not yet proven that $\|A\|_1$ decays *to zero*, even in the absence of $(v_\phi \boldsymbol{B}_\mathrm{m} A)$. Furthermore, a decay rate for $\|A\|_1$ is not generally determinable from (5.23) because the right side of (5.23) contains only surface and line integrals of $|A|$, rather than volume integrals, which would permit the use of variational inequalities.

5.5. *A scenario for decay to zero of* $\|A\|_1$ *when* $(v_\phi \boldsymbol{B}_\mathrm{m} A) = 0$

We conclude our discussion of A by outlining a 'proof' of the decay of $\|A\|_1$ to zero, which relies on two assumptions, one of which seems reasonable and the other possibly so. We adopt a factorization method similar to that used by Picard (see Courant & Hilbert (1962), p. 322), and in generalized maximum principles (P.W., p. 8), and which is the foundation of the Lortz & Meyer-Spasche derivation of (3.4). So let $A = fg$ and choose

$$(\mathcal{Q} + c) f = 0. \tag{5.28}$$

Then (2.23) is satisfied when $(v_\phi \boldsymbol{B}_\mathrm{m} A) = 0$, if

$$(\mathcal{Q} + 2\eta \nabla \ln f \cdot \nabla)\, g = 0. \tag{5.29}$$

While $V(t)$ may be quite general as in §2.1 (i), we add one additional constraint: that $\partial(\ln \varpi)/\partial n$ (or the appropriate limit as $\varpi \to 0$) is bounded on S. This only excludes those V that meet the z-axis tangentially (for example the torus $(\varpi - \varpi_0)^2 + z^2 = \varpi_0^2$), and is therefore not physically restrictive. Choose a sufficiently smooth positive function $h^2 \geqslant -2\partial(\ln \varpi)/\partial n$, and suppose, denoting the volume $\|1\|_1$ by $|V|$, that

$$\frac{\partial f}{\partial n} = -\left(h^2 + \frac{2}{\varpi}\frac{\partial \varpi}{\partial n}\right) f \quad \text{on } S, \tag{5.30}$$

$$f = 1/|V| \quad \text{at} \quad t = 0. \tag{5.31}$$

By application of maximum principles as in §5.2, $f \exp(-c_\mathrm{s} t)$ cannot have a non-positive space–time minimum (*a*) inside $V(t)$ for $t > 0$ by (5.28); (*b*) on S for $t > 0$ by (5.30); or (*c*) at $t = 0$ by (5.31). Thus f is positive for all $t \geqslant 0$. The ability to choose non-zero h in (5.30) is generally essential for part (*b*) of the argument, to ensure that the right side of (5.30) is non-negative when f is non-positive. Equation (5.30) is a simple extension of the boundary condition used by Lortz & Meyer-Spasche (1982*b*), who omit h, making their derivation of (3.4) not valid for all V 'topologically equivalent to a ball or torus' (sic), but rather for convex

large h^2 is chosen in (5.30). Other details concerning the derivation of (3.4) are essentially as
V where $\partial\varpi/\partial n \geqslant 0$ everywhere on S. However, (3.4) is more generally valid if a sufficiently
in Lortz & Meyer-Spasche (1982*b*) and are omitted here. In this section we are considering
a much stronger result than (3.4).

Using (5.30) and the divergence form of (5.28) (see (2.23) and (2.26)), we calculate the total rate of change of $\|f\|_1$ allowing for the motion of S:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V f\,\mathrm{d}V = \int_V \nabla\cdot\left\{\frac{\eta}{\varpi^2}\nabla(\varpi^2 f)\right\}\mathrm{d}V, \tag{5.32}$$

$$= -4\pi\int_I \eta f\,\mathrm{d}z - \int_S h^2\eta f\,\mathrm{d}S, \tag{5.33}$$

where $I = V\cap\{\varpi = 0\}$, as in §5.4. It is essential in (5.32) that the motion of S be identified with $\boldsymbol{v}$, and hence the results in this section do not apply to the Nernst–Ettingshausen effect: see §6, G.6.

From (5.33) we can conclude $\mathrm{d}\|f\|_1/\mathrm{d}t < 0$ and hence $\|f\|_1 \leqslant 1$, but not that $\|f\|_1 \to 0$ as $t\to\infty$. However, turning our attention to (5.29), the absence of a source term associated with c gives some hope of proving decay of g to zero as was done for χ in §4. We thus state a comparison theorem for g.

Theorem. *If, with* $\mathscr{Q}$ *as in* (2.24) *and* f *determined by* (5.28), (5.30), (5.31),

$$\{\mathscr{Q} + 2\eta\nabla\ln f\cdot\nabla\}\,w \leqslant 0 \quad \text{in } V,$$

$$w \geqslant |g| \quad \text{at } t = 0,$$

$$\partial w/\partial\varpi \leqslant 0 \quad \text{as } \varpi\to 0,$$

$$w \geqslant 0 \quad \text{on } S,$$

then

$$|g| \leqslant w \quad \text{in } V(t), \quad t\geqslant 0.$$

The proof is omitted, but is similar to and simpler than the proof of the comparison theorem for A in §5.2. To construct a comparison function for g consider, analogous to (5.8),

$$w = \beta(0)\,G(\varpi,\kappa(t))\,\mathrm{e}^{-qt},$$

where $\beta(0) = \max\{|g|/G(\varpi,\kappa(t)\}$ at $t = 0$, and proceed as in §5.3, but with $c = C = 0$, and equation (5.9*b*) replaced by the inequality

$$\kappa(t) \geqslant \sup_{V,\,t\geqslant 0}\left\{\frac{Rv_\varpi}{\eta} - \frac{\partial(\ln\eta)}{\partial\varpi}\right\} - \min_{V(t)}\left\{2\frac{\partial(\ln f)}{\partial\varpi}\right\}. \tag{5.34}$$

We again choose G as in (5.13), but with κ replaced by $\kappa(t)$. The possible time-dependence of κ stems from the presence of $\partial(\ln f)/\partial\varpi$ in (5.34), but we can choose κ so that both κ and $\mathrm{d}\kappa/\mathrm{d}t \geqslant 0$. Then (5.14*b*) simplifies to $q = \lambda/G(0)$ and any $\lambda > 0$ gives decay. Indeed, noting (5.31), we obtain, analogous to (5.20),

$$|g| \leqslant \begin{cases} \mathscr{A}(0)\,E(t;\tau_1)/|V|, & (5.35a)\\ \beta_1(0)\,\tau_1\,\mathrm{e}^{-t/\tau_1}, & (5.35b)\end{cases}$$

where $\beta_1(t) = \max\{|g|/G_1(\varpi,\kappa(t))\}$, analogous to (5.20), and where τ_1 is given by (5.17), with

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κ replaced by $\kappa(t)$. A sufficient condition for uniform decay of g to zero is $\tau_1/t \to 0$ as $t \to \infty$, i.e.

$$\frac{1}{t}\int_0^1 \int_0^\xi \exp\{\kappa(t)(\xi-\rho)\}\left(\frac{\rho}{\xi}\right)^3 \mathrm{d}\rho\,\mathrm{d}\xi \to 0. \tag{5.36}$$

Since $0 \leqslant \xi-\rho \leqslant 1$, a sufficient condition for (5.36) is $\kappa(t) \leqslant \ln t + \text{constant}$, or equally sufficient by (5.34)

$$2\frac{\partial(\ln f)}{\partial \varpi} \geqslant \ln(k/t) \quad \text{for large } t, \tag{5.37}$$

for any constant $k > 0$. Since

$$\|A\|_1 \leqslant \max_{V(t)} |g| \cdot \|f\|_1 \leqslant \max_{V(t)} |g|,$$

it then follows that $\|A\|_1 \to 0$ as $t \to \infty$, under the bounding curves in (5.35).

The two possible pitfalls of the preceding discussion are the assumption (*a*) that f exists satisfying (5.28), (5.30), (5.31), and (*b*) that f also satisfies (5.37). Consider these in turn.

(*a*) Existence of f is assured when S is stationary (Friedman 1964, p. 144) and for some moving boundaries (Friedman 1964, ch. 8). It seems reasonable to assume that f exists provided S changes sufficiently smoothly and perhaps slowly.

(*b*) Inequality (5.37) does not preclude steep gradients, but asks that negative ϖ-gradients do not develop faster than $\frac{1}{2}\ln(k/t)$ for large t. One might simply adopt (5.37) as a smoothness assumption, but it would be clearly preferable to derive (5.37) from the differential equation for f, and further work is needed in this regard. On some occasions it may be permissible to expand

$$f(\boldsymbol{r}, t) = \sum_i a_i f_i(\boldsymbol{r}) \exp(-\Omega_i t), \tag{5.38}$$

where Ω_0 is selected so that $\mathrm{Re}\{\Omega_0\} = \min \mathrm{Re}\{\Omega_i\}$. The general validity of (5.38) seems mathematically difficult to determine, requiring spectral analysis of the non-self-adjoint operator $\mathcal{Q}+c$. However, expansions such as (5.38) are commonly employed (see, for example, the mean field dynamos examined by Roberts (1972)) in numerical solutions of the induction equation when $\boldsymbol{v}$ and η are time-independent. In such cases, (5.37) is clearly satisfied since as $t \to \infty$, $\partial(\ln f)/\partial\varpi \to \partial(\ln f_0)/\partial\varpi$, which is determined by the initial conditions and hence bounded.

6. Generalizations and extensions

G.1. *Variable permeability*

Since variable μ is of little if any astroplanetary interest we omit most of the details and state only the necessary modifications to be made to the results given previously for μ_0.

Suppose that μ is axisymmetric and differentiable in V, $\hat{V}$, but possibly discontinuous across S. For the meridional field, the operators (2.18) and (2.20) are replaced by

$$\mathscr{P} \equiv \eta\nabla^2 - \frac{2\eta}{\varpi}\nabla\varpi\cdot\nabla - \eta\nabla(\ln\mu)\cdot\nabla - R\boldsymbol{v}\cdot\nabla - \frac{\partial}{\partial t},$$

$$\mathscr{E} \equiv \nabla^2 - \frac{2}{\varpi}\nabla\varpi\cdot\nabla - \nabla(\ln\mu)\cdot\nabla.$$

If we assume no surface currents, the standard magnetic boundary conditions are continuity of the normal component of $\boldsymbol{B}_\mathrm{m}$ and tangential component of $\boldsymbol{B}_\mathrm{m}/\mu$. Equivalently, χ and

$\mu^{-1}\,\partial\chi/\partial n$ are continuous across S. If we temporarily denote values on the inner and outer faces of S by subscripts $-$ and $+$, respectively, the comparison theorem of §4 is applicable under extension E.3 if $\partial(\chi-u)_-/\partial n < 0$ on any 'neutral' line N_0, which is on S. But, by the continuity of $\mu^{-1}\,\partial\chi/\partial n$

$$\frac{\partial}{\partial n}(\chi-u)_- = \frac{\mu_-}{\mu_+}\frac{\partial}{\partial n}(\chi-u)_+ + \frac{\mu_-}{\mu_+}\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n}.$$

By the outward derivative theorem used in §4.1 (ii) (P.W., theorem 7, p. 65), $\partial(\chi-u)_+/\partial n < 0$ on any N_0 that is on S (assuming $\chi - u \not\equiv \mathscr{M}$ in $\hat{V}$); and therefore $\partial(\chi-u)_-/\partial n < 0$ on such an N_0 if

$$\frac{1}{\mu_+}\frac{\partial u_+}{\partial n} \leqslant \frac{1}{\mu_-}\frac{\partial u_-}{\partial n}. \tag{6.1}$$

Inequality (6.1) is simply satisfied in several physically relevant circumstances.

(*a*) When μ is positive and continuous, any of the comparison functions of §4.2 may be used since they have the property that $\partial u/\partial n$ is continuous.

(*b*) For a sphere of fixed boundary $r = 1$, the spherically symmetric comparison functions mentioned at the end of §4.2 are applicable since they have the property $\partial u_+/\partial r = \partial u_-/\partial r = 0$ at $r = 1$.

For these cases uniform decay to zero of χ follows with (4.14*c*) replaced by

$$\kappa = \max\left[\sup_{\hat{V},t}\left\{-\frac{\partial(\ln\mu)}{\partial\varpi}\right\}, \sup_{V,t}\left\{-\frac{\partial(\ln\mu)}{\partial\varpi} - \frac{Rv_\varpi}{\eta}\right\}, 0\right].$$

Thus variable μ cannot prevent decay of the meridional field, but does influence the decay rate bound.

Modifications for variable permeability for A are slight. The comparison theorem and function of §5.2 and §5.3, and condition (5.16) for decay to zero of A, remain unchanged except that $\mathscr{Q}$, c, κ take the new forms

$$\mathscr{Q} \equiv \eta\nabla^2 + \left(\frac{2\eta}{\varpi}\boldsymbol{e}_\varpi + \nabla\eta - \eta\nabla\ln\mu - R\boldsymbol{v}\right)\cdot\nabla - \frac{\partial}{\partial t}, \tag{6.2a}$$

$$c = \frac{2}{\varpi}\frac{\partial\eta}{\partial\varpi} - \nabla\cdot(\eta\nabla\ln\mu) - R\nabla\cdot\boldsymbol{v}, \tag{6.2b}$$

$$\kappa = \sup_{V,t\geqslant 0}\left\{\frac{Rv_\varpi}{\eta} - \frac{\partial(\ln\eta)}{\partial\varpi} + \frac{\partial(\ln\mu)}{\partial\varpi}, 0\right\}. \tag{6.2c}$$

Furthermore, the operator $\mathscr{Q}+c$ retains its divergence form, the left side of (2.26) being altered by addition of only one term $-\nabla\cdot(\eta A\nabla\ln\mu)$, which does not contribute to the integral results (5.22) and (5.33). Thus the boundedness and decay results for $\|A\|_1$ in §5.4 and §5.5, remain unchanged apart from a minor modification to $\kappa(t)$ in (5.34) consistent with (6.2).

G.2. *Inner core and mantle*

Suppose that V contains a conducting sub-volume V_c, surface S_c, across which $\boldsymbol{v}$, σ are possibly discontinuous. The comparison theorem for χ in §4.1 then remains valid, although a new proof is required (similar to §5.2 for A), with consideration of a space–time maximum of $\pm\chi - u$ rather than just the space maximum. In particular, an outward derivative theorem for parabolic differential inequalities (P.W., theorem 6, p. 174) is needed to preclude a non-negative such maximum occurring on S_c. (The P.W. theorem must be extended somewhat (see Ivers 1984) to preclude the maximum occurring on S_c at $t = T$, the forward time boundary.) So none of the results for χ in §4 are altered by the presence of V_c (be it solid or liquid). The methods of §5 for A require more attention because $\partial A/\partial n$ (unlike $\partial\chi/\partial n$) is not generally continuous at discontinuities (such as S_c). The comparison theorem for A in §5.2 may be modified but loses its practicality in that it is no longer generally easy to construct comparison functions as in §5.3. However, the continuity across S_c of the tangential components of $\boldsymbol{E}$ ensures (via the Ohm law (2.8) and axisymmetry) the continuity of

$$\boldsymbol{n}\cdot\left\{\frac{1}{\sigma\varpi^2}\nabla\left(\frac{\varpi^2 A}{\mu}\right)-R\boldsymbol{v}A+\frac{Rv_\phi}{\varpi}\boldsymbol{B}_{\mathrm{m}}\right\}. \tag{6.3}$$

This, in turn, ensures that no contribution from S_c enters (5.22); and thus the conclusions for the boundedness and decay of $\|A\|_1$ in §5.4 remain intact, indeed even if μ is variable and discontinuous across S_c. All of the preceding discussion applies to any number of conducting sub-volumes and, in particular, to conducting mantles and cores.

G.3. *V and $\hat{V}$ multiply-connected or disconnected*

Bondi & Gold (1950) argued (not assuming axisymmetry) that if V was perfectly conducting ($\eta \equiv 0$) and $\hat{V}$ 'multiply connected, an external field can be caused to grow without limit'. Contrary to this, Moffatt (1978, §6.3) has emphasized the vital role of diffusion ($\eta > 0$) for sustained increase in the external dipole moment of $\boldsymbol{B}$ to occur; and Backus (1957, p. 521) has remarked that his pointwise non-amplification and integral-decay results for χ (see §3 herein) for the axisymmetric dynamo have 'nothing to do with connectivity'. None of the proofs of §4 and §5 require V and $\hat{V}$ to be simply connected. They are valid when V is a torus, for example. But we did assume (§2.1 (i)) that V and $\hat{V}$ were pathwise connected. Connectedness of V was for simplicity only, all of the arguments herein apply equally well to each connected component of a disconnected V. However, connectedness of $\hat{V}$ and the boundedness V were together a relatively unrestrictive means of ensuring that $\hat{V}$ cut the z-axis. This, in turn, justified argument (i) in the comparison theorem for χ in §4.1 on which the main aspects of §4 rely, and ensured (2.27), that $A \equiv 0$ in $\hat{V}$, on which the main aspects of §5 rely. Consider the alternative, that $\hat{V}$ is disconnected with a component $\hat{W}$ not intersecting $\{\varpi = 0\}$: for example, a non-conducting toroidal cavity inside V. The comparison theorem for χ, and hence all of §4, remains valid regardless; but, again, the comparison theorem requires a space–time maximum approach as in G.2. Also, as in G.2, the comparison theorem of §5.2 may be modified but becomes impractical. However, by associating (6.3) in V with a tangential component of $\boldsymbol{E}$, which is continuous on the boundary of $\hat{W}$, the integral results for A in §5.4 can be shown to remain unchanged (even if μ is variable and discontinuous), except that $\|A\|_1$ is interpreted as the integral of $|A|$ over $V \cup \hat{W}$ rather than just V.

G.4. *Conditional decay of B_ϕ, χ/ϖ, and other parameters*

One obvious question is whether or not the decay condition (5.16) for A can be removed or relaxed by considering the induction equation for some alternative representation of the azimuthal field, for example B_ϕ. The induction equation for B_ϕ is

$$\eta\nabla^2 B_\phi+\nabla\eta\cdot\nabla B_\phi-R\boldsymbol{v}\cdot\nabla B_\phi+cB_\phi-\partial B_\phi/\partial t=-R\varpi\boldsymbol{B}_{\mathrm{m}}\cdot\nabla(v_\phi/\varpi), \tag{6.4}$$

where
$$c = \frac{\partial}{\partial \varpi}\left(\frac{\eta}{\varpi}\right) - R\nabla\cdot\boldsymbol{v} + R\frac{v_\varpi}{\varpi}.$$

A comparison theorem similar to that of §5.2 is readily established with (5.5) replaced by the condition $w \geqslant 0$ on $\varpi = 0$; and comparison functions follow similar to those for A in §5.3. But clearly, owing to the presence of the undifferentiated term cB_ϕ in (6.4), decay of B_ϕ only results contingent on a condition similar to (5.16). This new condition is just as restrictive as (5.16) and thus no significant advantage is achieved. One may also contemplate induction equations for other representations of the azimuthal field, for example $f(\varpi)\,B_\phi$, and so generate a wide class of conditional decay results. However, it does not appear possible to remove the need for a condition like (5.16) by mere substitution. One could, of course, also consider alternative functional representations of the meridional field, for example χ/ϖ, but then generally to some disadvantage; the unconditional decay of χ in §4 being replaced by conditional decay, as in §5 for A.

G.5. *Charged exterior*

The exterior of V may carry a steady axisymmetric electric charge density ρ_e, all results remaining unaltered. The electric field $\boldsymbol{E}$ is determined in $\hat{V}$ from (2.7*b*), up to the addition of a scalar potential Φ, which is subsequently determined from $\nabla\cdot\boldsymbol{E} = \rho_e/\epsilon_0$, where ϵ_0 is the vacuum permittivity, and specification of the total charge in V_∞.

G.6. *The Nernst–Ettingshausen and similar effects*

Hibberd (1979) proposed a geomagnetic field model generated by the Nernst–Ettingshausen thermomagnetic effect as an alternative to a dynamo mechanism, and argued for long-term maintenance against ohmic dissipation. For Hibberd's specific model the poloidal field can be shown to decay to zero with a decay time no greater than $5\tau_{pol}$ (Ivers & James 1981). Generally the Nernst–Ettingshausen effect is mathematically similar to a dynamo, the induction equation (2.9) being altered only in that $\boldsymbol{v}$ is replaced by a thermomagnetic vector $\boldsymbol{G}$ parallel or antiparallel to the heat flow. On the other hand, $\boldsymbol{G}$ does not generally obey the boundary condition on $\boldsymbol{v}$ that represents no flow across S, so that a.d.ts cannot necessarily be directly translated to the Nernst–Ettingshausen effect. However, with the exception of §5.5, and one of the spherically symmetric comparison functions briefly mentioned at the end of §4.2.3, our results do not rely on the velocity boundary condition. Thus, the pointwise decay to zero of χ, the conditional pointwise decay of A to zero, and the strictly monotonic decay of $\|A\|_1$, all apply to an axisymmetric Nernst–Ettingshausen effect or equivalent mechanism.

7. Summary

In §3 of this paper we critically reviewed axisymmetric, mainly non-steady, antidynamo results. The strongest previously known results are for an incompressible uniform fluid ($\nabla\cdot\boldsymbol{v} = 0$; μ, σ constants), where it has definitely been established (Backus 1957) (i) that $\|\chi\|_2$ (defined in (3.3)) decays to zero with decay-time bound $4\tau_{pol} = 4/\pi^2$ and (ii) (with neglect of the generation of A from χ by azimuthal shearing (i.e. setting $(v_\phi \boldsymbol{B}_m A) \equiv 0$)) that $\mathscr{A}(t)$ (i.e. max $|A|$ at time t) decays to zero with decay-time bound τ_{pol}.

The bulk of this paper is concerned with much more general circumstances that allow compressible flow, variable conductivity σ, and moving boundary. In these circumstances established results are weaker. It has been known for some time (Backus 1957) that $X(t)$ (i.e. $\max|\chi|$ at t) could not increase; and the strongest mathematical result not restricting $\nabla\cdot\boldsymbol{v}$ or σ is that $X(t)$ must decrease strictly monotonically, but not necessarily to zero (Lortz & Meyer-Spasche 1982). To strengthen these results we have used maximum principles for elliptic and parabolic differential inequalities to prove a comparison theorem (§4.1). For our purposes this theorem replaces the difficult problem of considering the *equations* that determine χ (the induction equation (2.17) $\mathscr{P}\chi = 0$ in V, the current-free condition (2.19) $\mathscr{E}\chi = 0$ in $\hat{V}$, and boundary conditions) by the much simpler problem of solving corresponding *inequalities* ($\mathscr{P}u \leqslant 0$, $\mathscr{E}u \leqslant 0$, and boundary inequalities) for a comparison function u. The comparison theorem states $|\chi| \leqslant u$; and we have demonstrated (§4.2) a systematic way of constructing comparison functions u that decay exponentially to zero; all this not conditional on $\nabla\cdot\boldsymbol{v}$, variations in σ, or moving boundaries. It follows that $|\chi|$ decays uniformly to zero and $X(t)$ decays strictly monotonically to zero (equation (4.23)). Concomitantly, all L-norms $\|\chi\|_n$ and certain other field parameters, such as external multipole moments (§4.3) and net absolute surface flux (§4.5), decay to zero, but not necessarily monotonically. It must be realized that various senses of decay may be contemplated and decay of one field parameter does not always or simply imply decay of others. In particular, we have been forced to use considerable further analysis involving Schauder-type *a priori* estimates to prove that the meridional induction vector $\boldsymbol{B}_{\mathrm{m}}$ decays to zero, and then the decay is not shown to be uniform in space (equations (4.40), (4.41)). While this effectively removes the possibility of spiky behaviour in $\boldsymbol{B}_{\mathrm{m}}$ for large t, as contemplated by some authors (see the concluding comments in §4.4), the possible non-uniformness of the decay prevents concrete mathematical conclusions about unconditional decay of integrals involving $\boldsymbol{B}_{\mathrm{m}}$, such as the internal magnetic energy $\|\boldsymbol{B}_{\mathrm{m}}\|_2/2\mu$. Proof of unconditional decay of the energy integral is thus an outstanding problem that, if possible, seems to require different analysis and which we defer to a future time.

For the azimuthal field in compressible non-uniform non-steady conditions, nothing substantial seems to have been previously established. Here we have proven by an integral method that $\|A\|_1$ decays strictly monotonically if generation by $(v_\phi \boldsymbol{B}_{\mathrm{m}} A)$ is absent. Otherwise, it seems that $\|A\|_1$ may increase to a value determined by $(v_\phi \boldsymbol{B}_{\mathrm{m}} A)$ before finally decaying, although this is not established rigorously. In the absence of $(v_\phi \boldsymbol{B}_{\mathrm{m}} A)$ we prove (§5.5) the decay of $\|A\|_1$ is to zero unless very steep gradients develop in $\ln|A|$ for large t (see inequality (5.37)), and we are unable to totally exclude the latter possibility. By a comparison function method (§5.2 and §5.3) we also establish decay to zero of $\mathscr{A}(t)$ conditional upon the diffusivity gradient and dilatation rate being sufficiently small (condition (5.16)), and assuming $(v_\phi \boldsymbol{B}_{\mathrm{m}} A)$ decays uniformly to zero (equation (5.1)).

As by-products, our results do prove the impossibility of steady axisymmetric compressible non-uniform dynamos. In such cases, our proofs offer small advantages over those of Lortz (1968). (See the discussion in §4.2.3 and §5.4.)

Possibly our most important conclusion is negative. That is, despite considerable effort, we have been unable to dismiss compressible axisymmetric dynamos as astrogeophysically irrelevant. We have not been able to prove *unconditional* decay (i.e. without constraints on the compressibility and conductivity) on timescales not greatly exceeding a diffusion time. For small 'magnetic Reynolds numbers' κ (equations (4.14*c*), the antecedent to (4.25) and (5.9*b*)), the decay times of the comparison functions are not greatly different (table 1) from a diffusion time (based on maximum radius of V and minimum diffusivity). But for $\kappa \gtrsim 10$, as is probably astrogeophysically relevant, the decay times of the comparison functions typically increase subexponentially (like $e^{\alpha\kappa}/\kappa^n$, where $\alpha = \frac{1}{2}, 1$; and $n = 2, 3, 5$: see equations (4.24*b*), (4.28) and Appendix A). The resulting extremely large decay-time bounds are very slack in special cases (for example, for incompressible uniform fluids). But the question remains open as to whether extremely slowly-decaying compressible axisymmetric dynamos do exist and, in particular, with compressibility of an astrogeophysically relevant magnitude. Investigations pertinent to this question seem warranted in several directions, most obviously: (*a*) numerical computations with physically plausible velocity models, for example, by extending the method of Bullard & Gellman (1954) to compressible flows; (*b*) determination of astroplanetary relevant constraints on compressibility with a view to application of conditional-decay theorems. (One arguably very important attempt in this direction has already been made by Backus (1957).)

In §6 we have considered various generalizations and extensions. While these may be of little astroplanetary interest, they serve at least to demonstrate the robustness of our main decay results. That is, that $X(t)$ decays strictly monotonically to zero and, when $(v_\phi \boldsymbol{B}_{\mathrm{m}} A) = 0$, $\|A\|_1$ decays strictly monotonically, even when the permeability μ is variable; in the presence of inner cores, mantles, or charged exterior, regardless of the connectivity of the conducting volume V; and if the velocity $\boldsymbol{v}$ is replaced by the Nernst–Ettingshausen effect.

Appendix A. Asymptotic expression for the decay time τ_1

Evaluation of the inner integral in (5.17) gives

$$\tau_1(\kappa) = \int_0^1 \left\{\frac{6}{\kappa^4\xi^3}(e^{\kappa\xi}-1) - \frac{6}{\kappa^3\xi^2} - \frac{3}{\kappa^2\xi} - \frac{1}{\kappa}\right\} d\xi$$

$$= \int_0^\epsilon \left\{\frac{6}{\kappa^4\xi^3}(e^{\kappa\xi}-1) - \frac{6}{\kappa^3\xi^2} - \frac{3}{\kappa^2\xi} - \frac{1}{\kappa}\right\} d\xi + \int_\epsilon^1 \frac{6e^{\kappa\xi}}{\kappa^4\xi^3}\, d\xi - \int_\epsilon^1 \left(\frac{6}{\kappa^4\xi^3} + \frac{6}{\kappa^3\xi^2} + \frac{3}{\kappa^2\xi} + \frac{1}{\kappa}\right) d\xi,$$

where $0 < \epsilon < 1$ and ϵ is fixed. The first integral is bounded by

$$\left\{\frac{6}{\kappa^4\epsilon^3}(e^{\kappa\epsilon}-1) - \frac{6}{\kappa^3\epsilon^2} - \frac{3}{\kappa^2\epsilon} - \frac{1}{\kappa}\right\} \epsilon,$$

which is $O(\kappa^{-4}\, e^{\kappa\epsilon})$, as $\kappa \to \infty$; and the third integral is bounded by

$$\left\{\frac{6}{\kappa^4\epsilon^3} + \frac{6}{\kappa^3\epsilon^2} + \frac{3}{\kappa^2\epsilon} + \frac{1}{\kappa}\right\}(1-\epsilon),$$

which is $O(\kappa^{-1})$. Finally, we transform the second integral, by using $\zeta = 1-\xi$, to the Laplace integral (Erdelyi (1956), §2.2)

$$\frac{6e^\kappa}{\kappa^4}\int_0^{1-\epsilon} (1-\zeta)^{-3}\, e^{-\kappa\zeta}\, d\zeta,$$

which is asymptotic to

$$\frac{6e^\kappa}{\kappa^4}\left(\frac{1}{\kappa} + \frac{3}{\kappa^2} + \dots\right).$$

This is the dominant contribution to $\tau_1(\kappa)$.

Appendix B. Bounds for the derivatives of χ

Here we establish decay to zero of the first and second order space derivatives of χ by using Schauder *a priori* estimates for elliptic and parabolic equations. ('*A priori* estimates' are so called because they can be derived from the differential equations even before existence of a solution is established.) We will avoid notationally complicated details by transcribing appropriate results from Gilbarg & Trudinger (1977) (hereinafter G.T.) and Friedman (1964) (hereinafter F.).

(a) *Interior estimates in* $\hat{V}$

Let $\Omega(t)$ be the space domain consisting of all cartesian points $\boldsymbol{x} = (x, y, z)$ in $\hat{V}(t)$ excluding the z-axis. Let $d_x = d$ be the distance from $\boldsymbol{x}$ to the boundary $\partial\Omega$, which consists of $S(t)$ and the z-axis. Let $d_{x_1 x_2} = \min\{d_{x_1}, d_{x_2}\}$, and let $D^j\chi(\boldsymbol{x})$ be any one of the jth order cartesian space derivatives of χ.

Theorem. *Given equation* (2.19) *in* Ω, *then*

$$\sum_{j\leqslant 2}\sup_{\Omega(t)}\{d^j\,|D^j\chi(\boldsymbol{x})|\}+\sup_{\Omega(t)}\left\{d^{2+\alpha}_{x_1x_2}\frac{|D^2\chi(\boldsymbol{x}_1)-D^2\chi(\boldsymbol{x}_2)|}{|\boldsymbol{x}_1-\boldsymbol{x}_2|^{\alpha}}\right\}\leqslant CX(t), \tag{B 1}$$

for all Hölder exponents α *where* $0<\alpha\leqslant 1$, *and constant* C *which depends only on* α, *not on* Ω *or* t. (*The summation in* (B 1) *is over all space derivatives of order less than or equal to two.*)

Proof. Since the coefficients of $\mathscr{E}$ in (2.20) are analytic in $\Omega(t)$, so is χ (G.T., theorem 6.17 and following remarks). Equation (B 1) then follows directly from theorem 6.2 of G.T. provided we can show that the Hölder norms

$$\left|\frac{2x}{x^2+y^2}\right|^{(1)}_{0,\alpha;\Omega},\quad \left|\frac{2y}{x^2+y^2}\right|^{(1)}_{0,\alpha;\Omega},$$

corresponding to the cartesian forms of the coefficients in (2.20), are bounded. By definition (G.T., equation (6.10))

$$\left|\frac{2x}{x^2+y^2}\right|^{(1)}_{0,\alpha;\Omega}=s_1+s_2,$$

where

$$s_1=\sup_{\Omega(t)}\left\{d\left|\frac{2x}{x^2+y^2}\right|\right\},$$

$$s_2=\sup_{\Omega(t)}\left\{d^{1+\alpha}_{x_1x_2}\,|\boldsymbol{x}_1-\boldsymbol{x}_2|^{-\alpha}\left|\frac{2x_1}{x_1^2+y_1^2}-\frac{2x_2}{x_2^2+y_2^2}\right|\right\}.$$

Now since

$$d\leqslant (x^2+y^2)^{\frac{1}{2}}, \tag{B 2}$$

then

$$s_1\leqslant\sup_{\Omega(t)}|2x(x^2+y^2)^{-\frac{1}{2}}|=2.$$

Also, since V is bounded, the equality in (B 2) is actually attained for points sufficiently close to the z-axis and far from S (for example $x\leqslant 1$, $y=0$, $z>2$). Thus $s_1=2$.

To evaluate s_2, introduce complex coordinates $\zeta_j = x_j + iy_j$ $(j = 1, 2)$, and suppose without loss of generality that $|\zeta_1| \geqslant |\zeta_2|$. Then $d_{x_1x_2} \leqslant |\zeta_2|$, and

$$s_2 \leqslant \sup_{\Omega(t)} \left\{ |\zeta_2|^{1+\alpha} |\zeta_1 - \zeta_2|^{-\alpha}\, 2 \left| \mathrm{Re}\left(\frac{1}{\zeta_1} - \frac{1}{\zeta_2}\right) \right| \right\}$$

$$\leqslant 2 \sup_{\Omega(t)} \{ |\zeta_1 - \zeta_2|^{1-\alpha} |\zeta_1^{-1} \zeta_2^{\alpha}| \} \leqslant 2^{2-\alpha}.$$

Again, this bound is actually attained for certain points sufficiently close to the z-axis and far from S (for example $x_1 = -x_2$, $|x| \leqslant 1$, $y_1 = y_2 = 0$, $z_1 = z_2$, $|z_1| > 2$). Thus $s_2 = 2^{2-\alpha}$, and

$$\left| \frac{2x}{x^2 + y^2} \right|^{(1)}_{0,\alpha;\Omega} = 2 + 2^{2-\alpha} \leqslant 6.$$

Finally, by the axisymmetry of Ω,

$$\left| \frac{2y}{x^2 + y^2} \right|^{(1)}_{0,\alpha;\Omega} = \left| \frac{2x}{x^2 + y^2} \right|^{(1)}_{0,\alpha;\Omega}.$$

Thus the relevant Hölder norms are bounded, theorem 6.2 of G.T. applies, and (B 1) follows.

COROLLARIES. *By selecting appropriate single terms from the left side of* (B 1), *we obtain the bounds*

$$d|\nabla\chi| \leqslant CX(t), \quad d^2|\nabla^2\chi| \leqslant CX(t).$$

Consequently
$$|\boldsymbol{B}| = \frac{1}{\varpi}|\nabla\chi| \leqslant \frac{C}{\varpi d} X(t),$$

which establishes (4.40).

(*b*) *Interior estimates in V*

Here let Ω be the space–time domain of all cartesian points $P(\boldsymbol{x}, t)$ such that $\boldsymbol{x}$ lies in $V(t)$ excluding the z-axis and $0 \leqslant t \leqslant T$. Let $\Omega^{\bullet}$be the exterior of Ω, and let $\partial\Omega$ be that part of the boundary of Ω composed of $V(0)$, the z-axis, and all surfaces $S(t')$ for $0 \leqslant t' \leqslant t$. Define the parabolic 'distance' between the space–time points P_1, P_2 by

$$d(P_1, P_2) = \{(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 + |t_1 - t_2|\}^{\frac{1}{2}}. \tag{B 3}$$

Let d_P be the minimum 'distance' from P to $\partial\Omega$, and $d_{P_1P_2} = \min\{d_{P_1}, d_{P_2}\}$. Since $r \leqslant 1$ in Ω, $d(P_1, P_2) \leqslant (4 + T)^{\frac{1}{2}}$, $d_P \leqslant \frac{1}{2}$, and $d_{P_1P_2} \leqslant \frac{1}{2}$. Lett D^j represent space derivatives as in part (*a*).

THEOREM. *Given* (2.17) *in* Ω *and conditions* (2.12)–(2.14) *on* η *and* $\boldsymbol{v}$, *then*

$$\sum_{j\leqslant 2} \left[\sup_{\Omega} \{d_P^j |D^j\chi(P)|\} + \sup_{\Omega} \left\{ d_{P_1P_2}^{\alpha+j} \frac{|D^j\chi(P_1) - D^j\chi(P_2)|}{d(P_1, P_2)^{\alpha}} \right\} \right]$$
$$+ \sup_{\Omega} \left\{ d_P^2 \left| \frac{\partial\chi}{\partial t} \right| \right\} + \sup_{\Omega} \left\{ d_{P_1P_2}^{\alpha+2} \frac{|\partial\chi(P_1)/\partial t - \partial\chi(P_2)/\partial t|}{d(P_1, P_2)^{\alpha}} \right\} \leqslant KX(0), \tag{B 4}$$

for all Hölder exponents α *where* $0 < \alpha < 1$, *and constant* K *which depends only on* α, T, η_0, η_1, K_1, K_2, K_5 *and* K_6 (*from* (2.12), (2.14)).

Proof. (B 4) is a specific application of results given by F. (theorem 5, ch. 3; theorem 1, ch. 4; extension of theorem 1, p. 128), provided the Hölder norms (F. equation (2.11), ch. 3)

$$|\eta|_\alpha, \quad |d\boldsymbol{v}|_\alpha, \quad \left|d\left(\frac{2x\eta}{x^2+y^2}\right)\right|_\alpha, \quad \left|d\left(\frac{2y\eta}{x^2+y^2}\right)\right|_\alpha,$$

corresponding to the coefficients in (2.18) are bounded. It is in bounding these norms that (2.12) and (2.14) are important. For example, by definition,

$$|\eta|_\alpha = \sup_\Omega |\eta| + s_3 = \eta_1/\eta_0 + s_3,$$

where

$$s_3 = \sup_\Omega \left\{d^\alpha_{P_1P_2}\frac{|\eta(P_1)-\eta(P_2)|}{d(P_1,P_2)^\alpha}\right\}.$$

The term s_3 requires careful attention; for, in general, it is quite possible that the space–time straight line P_1P_2 does not lie everywhere inside Ω. Thus, although η is differentiable inside Ω, it is not necessarily differentiable along P_1P_2. Correspondingly, it is quite possible to have P_1 and P_2 close together but separated by intervening parts of $\hat{\Omega}$. Then $|\eta(P_1)-\eta(P_2)|$ is not necessarily small and the boundedness of s_3 is questionable in the limit of P_1 being arbitrarily close to P_2. However, the weighting factor $d^\alpha_{P_1P_2}$ removes this possible unboundedness. For, if part of $\hat{\Omega}$ separates P_1 form P_2, then $d_{P_1P_2} \leqslant d(P_1,P_2)$, so that

$$s_3 \leqslant \sup_\Omega |\eta(P_1)-\eta(P_2)| \leqslant \eta_1/\eta_0 - 1.$$

Alternatively, if $d_{P_1P_2} \geqslant d(P_1,P_2)$, then the convexity of the hypersurface $d(P,P_1) = d(P_1,P_2)$ ensures that the straight line P_1P_2 lies entirely inside Ω. Then the mean-value theorem along P_1P_2 implies

$$\begin{aligned} s_3 &\leqslant \sup_\Omega \left\{\left|\left(\nabla\eta, \frac{\partial\eta}{\partial t}\right)\right| \cdot |P_1P_2|\, d(P_1,P_2)^{-\alpha}\right\} \\ &\leqslant (K_1+K_2)\begin{cases}|P_1P_2|^{1-\alpha}, & \text{if } |t_1-t_2| \leqslant 1; \\ d(P_1,P_2)^{1-\alpha}, & \text{if } |t_1-t_2| \geqslant 1,\end{cases} \\ &\leqslant (K_1+K_2)\,(4+T^2)^{\frac{1}{2}}. \end{aligned}$$

(Here, $|P_1P_2| = \{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2+(t_1-t_2)^2\}^{\frac{1}{2}}$ as usual.) Thus

$$|\eta|_\alpha \leqslant \eta_1/\eta_0 + \max\{\eta_1/\eta_0 - 1,\ (K_1+K_2)\,(4+T^2)^{\frac{1}{2}}\}.$$

Similarly,

$$\begin{aligned} |dv_i|_\alpha &= \sup_\Omega \{d_P|v_i|\} + \sup_\Omega \left\{d^{1+\alpha}_{P_1P_2}\frac{|v_i(P_1)-v_i(P_2)|}{d(P_1,P_2)^\alpha}\right\} \\ &\leqslant 1+\max\{2,\ (K_5+K_6)\,(4+T^2)^{\frac{1}{2}}\}. \end{aligned}$$

The two remaining Hölder norms can be bounded as follows:

$$\begin{aligned} \left|d\,\frac{2\eta x}{x^2+y^2}\right|_\alpha &= \left|d\frac{2\eta y}{x^2+y^2}\right|_\alpha, \quad \text{by axisymmetry,} \\ &\leqslant \sup_\Omega \{\eta\}\left|d\frac{2x}{x^2+y^2}\right|_\alpha, \quad \text{by axisymmetry and since } \eta > 0, \\ &\leqslant \frac{\eta_1}{\eta_0}(2+2^{2-\alpha}), \quad \text{from part } (a). \end{aligned}$$

(This result differs from (*a*) in that equality alone cannot generally be taken since points sufficiently close to the z-axis do not necessarily lie in V.)

Corollaries. *By translating the time span from* $[0, T]$ *to* $[t_0, t]$ *and redefining* Ω, $\partial\Omega$ *correspondingly*, (B 4) *remains valid with the right side replaced by* $KX(t_0)$, *where* K *now depends on* α, η_0, η_1, K_1, K_2, K_5, K_6, *and the span* $t-t_0$. *Then, by selecting appropriate terms from the left side of* (B 4),

$$d_P|\nabla\chi| \leqslant KX(t_0), \quad d_P^2|\nabla^2\chi| \leqslant KX(t_0),$$

implying

$$|\boldsymbol{B}_\mathrm{m}| \leqslant KX(t_0)/\varpi d_P, \tag{B 5}$$

and

$$|\mu_0 j_\phi| = \left|\nabla^2\chi - 2\frac{\nabla\varpi}{\varpi}\cdot\nabla\chi\right| \leqslant |\nabla^2\chi| + \frac{2}{\varpi}|\nabla\chi| \leqslant (1+2d_P)\frac{KX(t_0)}{\varpi d_P^2}. \tag{B 6}$$

Since $X(t)$ is monotonically decreasing, (B 5), (B 6) tend to be optimized by taking t_0 closer to t. To determine a convenient quasi-optimum t_0 let us briefly consider more closely the nature of d_P. Suppose that the smallest 'distance' from $P(\boldsymbol{x}, t)$ to the space boundary $S \cup \{\varpi = 0\}$ during (t_0, T) occurs at time t^* when the boundary point $\boldsymbol{x}^*$ happens to be spacewise closest to $\boldsymbol{x}$. Then

$$d_P = \min\,[\{(\boldsymbol{x}-\boldsymbol{x}^*)^2 + t - t^*\}^{\frac{1}{2}}, (t-t_0)^{\frac{1}{2}}]. \tag{B 7}$$

Clearly, letting $t_0 = t$ is no use, because then $d_P = 0$ and the right side of (B 5), (B 6) are infinite. On the other hand, since $d_P \leqslant \frac{1}{2}$, a simplification of d_P results from choosing $t - t_0 \geqslant \frac{1}{4}$. Then (B 7) reduces to

$$d_P = \{(\boldsymbol{x}-\boldsymbol{x}^*)^2 + t - t^*\}^{\frac{1}{2}};$$

and it follows that $d_P \geqslant |\boldsymbol{x}-\boldsymbol{x}^*| \geqslant d$, where d is the minimum space distance from $\boldsymbol{x}$ to the space boundary at any time during $[t_0, t]$. This result and the choice $t_0 = t - \frac{1}{4}$ results in (4.41) and (4.42).

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References

Abramowitz, M. & Stegun, I. A. 1972 *Handbook of mathematical functions*. New York: Dover Publications.
Backus, G. E. 1957 The axisymmetric self-excited fluid dynamo, *Astrophys. J.* **125**, 500–524.
Backus, G. E. 1968 Kinematics of geomagnetic secular variation in a perfectly conducting core. *Phil. Trans. R. Soc. Lond.* A **263**, 239–266.
Backus, G. E. & Chandrasekhar, S. 1956 On Cowling's theorem on the impossibility of self-maintained axisymmetric homogeneous dynamos. *Proc. natn Acad. Sci. U.S.A.* **42**, 105–109.
Bondi, H. & Gold, S. 1950 On the generation of magnetism by fluid motion. *Mon. Not. Roy. astr. Soc.* **110**, 607–611.
Booker, J. R. 1969 Geomagnetic data and core motions. *Proc. R. Soc. Lond.* A **309**, 27–40.

Braginskii, S. I. 1964 Self-excitation of a magnetic field during the motion of a highly conducting fluid. *Zh. éksp. teor. Fiz.* **47**, 1084–1098. Translated in *Soviet Phys. JETP* **20**, 726–735 (1965).
Bullard, E. C. & Gellman, H. 1954 Homogeneous dynamos and terrestrial magnetism. *Phil. Trans. R. Soc. Lond.* A **247**, 213–278.
Chandrasekhar, S. 1956 Effect of internal motions on the decay of a magnetic field in a fluid conductor. *Astrophys. J.* **124**, 244–265.
Chapman, S. & Bartels, J. 1962 *Geomagnetism*, vol. 2. Oxford: Clarendon Press.
Childress, S. 1969 *Théorie magnétohydrodynamique de l'effet dynamo. Rep. Dép. Méch. Fac. Sci.*, *Université de Paris.*
Courant, R. & Hilbert, D. 1962 *Methods of mathematical physics*, vol. 2. New York: Interscience.
Cowling, T. G. 1934 The magnetic field of sunspots. *Mon. Not. R. astr. Soc.* **94**, 39–48.
Cowling, T. G. 1955 Dynamo theories of cosmic magnetic fields. *Vistas Astr.* **1**, 313–322.
Cowling, T. G. 1957 The dynamo maintenance of steady magnetic fields. *Q. Jl Mech. appl. Math.* **10**, 129–136.
Erdelyi, A. 1956 *Asymptotic expansions*. New York: Dover Publications.
Friedman, A. 1964 *Partial differential equations of parabolic type*. Englewood Cliffs: Prentice-Hall.
Gilbarg, D. & Trudinger, N. S. 1977 *Elliptic partial differential equations of the second order*. Berlin: Springer-Verlag.
Hibberd, F. H. 1979 The origin of the Earth's magnetic field *Proc. R. Soc. Lond.* A **369**, 31–45.
Hide, R. 1978 How to locate the electrically conducting fluid core of a planet from external magnetic observations. *Nature, Lond.* **271**, 640–641.
Hide, R. 1979 Dynamo theorems. *Geophys. Astrophys. Fluid Dynamics* **14**, 183–186.
Hide, R. 1981 The magnetic flux linkage of a moving medium: A theorem and geophysical consequences. *J. geophys. Res.* **86**, (11) 681–687.
Hide, R. & Malin, S. R. C. 1979 The size of Jupiter's electrically conducting fluid core. *Nature, Lond.* **280**, 42–43.
Hide, R. & Palmer, T. N. 1982 Generalization of Cowling's theorem. *Geophys. Astrophys. Fluid Dynamics* **19**, 301–309.
Ivers, D. J. 1984 *Antidynamo theorems*, Ph.D. thesis, University of Sydney.
Ivers, D. J. & James, R. W. 1981 On the maintenance of magnetic fields by compressible flows and the Nernst–Ettingshausen effect, *Geophys. Astrophys. Fluid Dynamics*, **16**, 319–323.
Larmor, J. 1919 How could a rotating body such as the Sun become a magnet? *Rep. Br. Ass. Advmt Sci.* 159–160.
Lortz, D. 1968 Impossibility of steady dynamos with certain symmetries. *Physics Fluids* **11**, 913–916.
Lortz, D. & Meyer-Spasche, R. 1982*a* On the decay of symmetric dynamo fields. *Math. Meth. appl. Sci.* **4**, 91–97.
Lortz, D. & Meyer-Spasche, R. 1982*b* On the decay of symmetric toroidal dynamo fields. *Z. Naturf.* **37**a, 736–740.
Moffatt, H. K. 1978 *Generation of magnetic fields by electrically conducting fluids*. Cambridge University Press.
Parker, E. N. 1979 *Cosmical magnetic fields*. Oxford: Clarendon Press.
Protter, M. H. & Weinberger, H. F. 1967 *Maximum principles in differential equations*. Englewood Cliffs: Prentice-Hall.
Roberts, P. H. 1972 Kinematic dynamo models. *Phil. Trans. R. Soc. Lond.* A **272**, 663–698.
Todoeschuck, J. P. & Rochester, M. G. 1980 The effect of compressible flow on antidynamo theorems. *Nature, Lond.* **284**, 250–251.
Vekua, I. N. 1962 *Generalized analytic functions*. Oxford: Pergamon Press.